KRB
Maths Dept.

Year 12 - 3 Unit Mathematics

Assessment Task 2

Date: 15/5/95 TOPICS: INVERSE TRIG. FUNCTIONS Time Allowed: 45 Minutes

Question 1 (9 Marks)

Find the exact value of the following without the use of a calculator. Show all working.

a) $\cos^{-1}\left(\frac{-1}{2}\right)$ (2 marks)

b) $\cot\left(\sin^{-1}\frac{1}{3}\right)$ (3 marks)

c) $\cos\left(2\sin^{-1}\frac{7}{25}\right)$ (4 marks)

Question 2 (8 Marks)

Given $y = \sin^{-1}x$

a) Explain why $0 \le \cos y \le 1$ (3 marks)

b) Prove that $\cos y = \sqrt{1 - x^2}$ (3 marks)

c) Prove that $\frac{dx}{dy} = \cos y$ (2 marks)

Question 3 (6 Marks)

Differentiate the following with respect to x

a) $y = \left(\sin^{-1}3x\right)^4$ (3 marks)

b) $y = \log_e(x^2+1)\tan^{-1}x$ (3 marks)

Question 4 (5 Marks)

Find

a) $\int \frac{2dx}{\sqrt{4 - 25x^2}}$ (3 marks)

b) $\int \frac{dx}{4 + (x-1)^2}$ (2 marks)

Question 5 **(4 Marks)**

Given $\int_{-a}^{a} \frac{dx}{1+x^2} = \frac{\pi}{2}$ find the value of a

Question 6 **(5 Marks)**

Show that the equation of the normal to the curve $y = \tan^{-1} 5x$ at $x = \frac{1}{5}$ is $40x + 100y - 25\pi - 8 = 0$

Question 7 **(7 Marks)**

Given $y = |x - 2|$

a) Sketch $y = |x - 2|$ showing the important features (1 mark)

b) State the largest positive domain and the range such that the inverse function exists. (2 marks)

c) Using the domain of b) and the definition of the absolute value function, find the inverse function $f^{-1}(x)$
<u>Explain all steps</u> in your working. (4 marks)

Question 8 **(6 Marks)**

Which of the following statements is always true?

i) $\cos^{-1}(\cos x) = x$

ii) $\cos(\cos^{-1} x) = x$

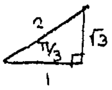
By considering the case $x = -a$ support your answer to the above

Year 12 (3U) - Assessment Task 2 - Inverse Function Solutions (50) 15/5/9

1/ (9)

a) $\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\left(\frac{1}{2}\right)$

$= \pi - \frac{\pi}{3}$ } (1)

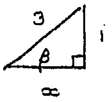


$= \frac{2\pi}{3}$ (1)

(2)

b) $\cot\left(\sin^{-1}\frac{1}{3}\right)$

let $\beta = \sin^{-1}\frac{1}{3}$ (1)



$x^2 + 1^2 = 3^2$

$x = \sqrt{8}$

$x = 2\sqrt{2}$ (1)

$\therefore \cot\left(\sin^{-1}\frac{1}{3}\right) = \cot\beta$

$= \frac{1}{\tan\beta}$

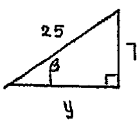
$= \frac{2\sqrt{2}}{1}$

$= 2\sqrt{2}$ (1)

(3)

c) $\cos\left(2\sin^{-1}\frac{7}{25}\right)$

let $\sin^{-1}\frac{7}{25} = \beta$ $\quad \sin\beta = \frac{7}{25}$



$y^2 + 7^2 = 25$

$y = 24$ (1)

$\cos\left(2\sin^{-1}\frac{7}{25}\right) = \cos(2\beta)$ (1)

$= \cos^2\beta - \sin^2\beta$ (1)

$= \left(\frac{24}{25}\right)^2 - \left(\frac{7}{25}\right)^2$

$= \frac{576 - 49}{625}$

$= \frac{527}{625}$ (1)

(4)

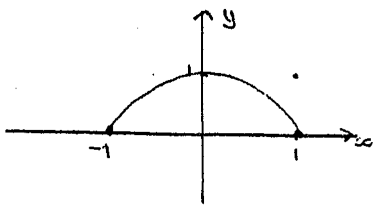
2/ $y = \sin^{-1}x$ (8)

a) $\cos y = \cos(\sin^{-1}x)$

now $-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$ (1)

$\cos\left(\frac{\pi}{2}\right) = 0$

$\cos\left(-\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$ } (1)



by symmetry

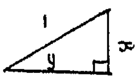
$\cos(0) = 1$ (1)

Hence $0 \leq \cos(\sin^{-1}x) \leq 1$

(3)

b) $y = \sin^{-1}x$

then $x = \sin y$ (1)



by Pythagoras third side

$= \sqrt{1 - x^2}$ (1)

Hence $\cos y = \frac{\sqrt{1-x^2}}{1}$

$\cos y = \sqrt{1 - x^2}$ } (1)

(3)

c) since $x = \sin y$ (1)

then $\frac{dx}{dy} = \cos y$

or $y = \sin^{-1}x$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\frac{dx}{dy} = \sqrt{1 - x^2}$

$\therefore \frac{dx}{dy} = \cos y$

3/ (6)

a) $y = (\sin^{-1}3x)^4$

$\frac{dy}{dx} = 4(\sin^{-1}3x)^3 \times \frac{1}{\sqrt{1-(3x)^2}} \times 3$ (1) (1)

$= \frac{12(\sin^{-1}3x)^3}{\sqrt{1 - 9x^2}}$ (1)

(

b) $y = \log_e(x^2+1)\tan^{-1}x$

$\frac{dy}{dx} = \frac{2x}{x^2+1} \times \tan^{-1}x$ (1) $+$

$\log_e(x^2+1) \times \frac{1}{1+x^2}$ (1)

(1)

$= \frac{2x\tan^{-1}x + \log_e(x^2+}{1+x^2}$

(3)

4/ a) $\int \frac{2}{\sqrt{4 - 25x^2}}$ (5)

$= \int \frac{2}{\sqrt{25\left(\frac{4}{25} - x^2\right)}} \, dx$

$= \int \frac{2}{5\sqrt{\frac{4}{25} - x^2}} \, dx$

$= \frac{2}{5} \int \frac{1}{\sqrt{\frac{4}{25} - x^2}} \, dx$ (1)

$= \frac{2}{5} \sin^{-1} \frac{x}{\frac{2}{5}} + c$

$= \frac{2}{5} \sin^{-1} \frac{5x}{2} + c$ (1) (1)

(3)

b) $\int \frac{dx}{4 + (x-1)^2}$

$= \frac{1}{2} \tan^{-1} \frac{(x-1)}{2} + c$

(2)

5/ (4)

$\int_{-a}^{a} \frac{dx}{1+x^2} = \frac{\pi}{2}$

$\left[\tan^{-1} x\right]_{-a}^{a} = \frac{\pi}{2}$ (1)

$\tan^{-1}(a) - \tan^{-1}(-a) = \frac{\pi}{2}$

$\tan^{-1}(a) - -\tan^{-1}(a) = \frac{\pi}{2}$ (1)

$2 \tan^{-1}(a) = \frac{\pi}{2}$

$\tan^{-1} a = \frac{\pi}{4}$ (1)

$\therefore a = 1$ (1)

6/ $y = \tan^{-1} 5x$ (5)

at $x = \frac{1}{5}$ $y = \tan^{-1} 1$

$y = \frac{\pi}{4}$ (1)

$\frac{dy}{dx} = \frac{1}{1 + (5x)^2} \times 5 = \frac{5}{1 + 25x^2}$ (1)

at $x = \frac{1}{5}$ $\frac{dy}{dx} = \frac{5}{1 + 25\left(\frac{1}{5}\right)^2}$

$= \frac{5}{1+1}$

$m_1 = \frac{5}{2}$

$\therefore$ grad $m_2 = \frac{-2}{5}$ (1) (normal) $m_1 \times m_2 = -1$

$y - \frac{\pi}{4} = \frac{-2}{5}\left(x - \frac{1}{5}\right)$ (1)

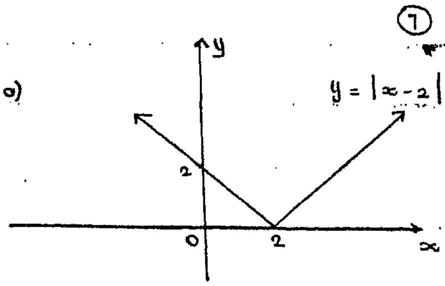
$5y - \frac{5\pi}{4} = -2x + \frac{2}{5}$

(×20) $100y - 25\pi = -40x + 8$

$40x + 100y - 25\pi - 8 = 0$ (1)

7/ (7)

a)



(1)

b) domain $\{x : x \geq 2 ; x \in R\}$ (1)

range $\{y : y \geq 0 \ y \in R\}$ (1)

c) $y = |x - 2|$

Inverse interchanging domain & range

so r: $y \geq 2$ d: $x \geq 0$

$x = |y - 2|$

now (1)

$x = y - 2$ if $y - 2 \geq$

ie y

$= -(y - 2)$ if $y <$ (1)

We choose

$x = y - 2$ if $y \geq 2$

$\therefore x = y - 2$ $y \geq 2$

or $y = x + 2$ for x

$\therefore f^{-1}(x) = x + 2$ $(x \geq$ (1)

8/ (6)

$\cos(\cos^{-1} x) = x$ is alw. true (1)

consider $x = -a$

(i) $\cos^{-1}(\cos(-a))$

$= \cos^{-1}(+\cos a)$ (1)

$= \cos^{-1}(\cos a)$

$= a$ (1)

$\neq -a$

(ii) $\cos\left(\cos^{-1}(-a)\right)$

$= \cos\left(\pi - \cos^{-1} a\right)$ (1)

$= -\cos\left(\cos^{-1} a\right)$ (1)

2nd quad

$= -a$ (1)

$\therefore \cos\left(\cos^{-1}(-a)\right) = -a$ ✓

Hence only (ii) is true.