MOUNT MORIAH COLLEGE

TO LEARN TO ACT TO HEED

2009

ASSESSMENT TASK 5 EXAMINATION

14th SEPTEMBER 2009

MATHEMATICS EXTENSION 1

Student Number:

Time allowed: 50 minutes

- Calculators may be used.
- Show all necessary working
- Start **each question** on a new page.

TOTAL /37

Question 1 (13 Marks) Start each question on a new page

Marks

(a) Four girls and four boys are to stand in a row. Find the number of arrangements if:

(i) there are no restrictions. 1

(ii) the girls want to stand together in a group. 1

(iii) boys and girls alternate. 1

(iv) two boys want to stand together. 1

(v) a particular girl wants to stand in front of the row. 1

(vi) John and Sally want to be together and the boys and girls alternate. 1

(vii) John and Sally must be separated by at least two persons. 1

(b) (i) The letters of the word SQUARE are arranged at random in a straight line. Find:

(α) the total number of possible arrangements. 1

(β) the probability all three vowels are next to each other. 1

(γ) the probability no two vowels are next to each other. 1

(ii) The letters of the word SQUARE are arranged at random in a circle. Find:

(α) the total number of possible arrangements. 1

(β) the probability all three vowels are next to each other. 1

(γ) the probability no two vowels are next to each other. 1

(c) If the letters of MATHEMATICS are used to form a word, taken all at a time, how many distinct words can be formed?

(i) Of these words, how many start with M and end with S? 1

(ii) If a word is selected at random find the probability that the 2 M's stay together. 1

(iii) If a word is selected at random find the probability that the letters T,H, E stay together. 1

Question 2:(10 Marks) Start each question on a new page

Marks

(a) How many ways can a committee of three persons be selected from 4 married couples if

(i) all are equally eligible. 1

(ii) the committee must consist of two men and a woman. 1

(iii) the committee must consist of at least one woman. 1

(iv) husband and wife cannot serve in the same committee. 1

(b) A student takes a test with 7 questions and guesses on each question. If the probability of guessing the correct answer is 0.2 on each question, calculate that the student answers:

(a) exactly 4 questions correctly. 1

(b) at least one question correctly. 1

(c) What is the most likely number of questions answered correctly. (Greatest term in expansion $(p+q)^k$) 1

(c) A die is biased so that in 10 throws the probability that 6 occurs exactly three times is three quarters the probability that 6 occurs exactly once. Find the probability that 6 occurs on a single roll of the die. 3

Question 3:(14 Marks) Start each question on a new page

Marks

(a) Find the term independent of x in the expansion of $\left(x-\frac{1}{2x^3}\right)^{20}$. 2

(b) Find the coefficient of x^2 in the expansion of $\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)^9$. 3

(c) By considering that $(1+x)^{2n}=(1+x)^n(1+x)^n$ and by examining the coefficient of x^n on each side of the identity, show that ${}^{2n}C_n=\sum_{k=0}^{n}\left({}^nC_k\right)^2$. 3

(d) By treating $(1-2x+x^2)$ as $1-x(2-x)$ and hence as $(1-a)$, where $a=x(2-x)$, find the first four terms of $(1-2x+x^2)^4$. 3

(e) Using the identity $(1+x)^n=\sum_{r=0}^{n}{}^nC_r x^r$, and by integrating with respect to x show that $\frac{1}{n+1}\left(2^{n+1}-1\right)=\sum_{r=0}^{n}\frac{1}{r+1}\,{}^nC_r$ 3

Question 1:

(a) (i) $8! = 40320$.

(ii) $4! \times 5! = 2880$

(iii) $4! \times 4! \times 2 = 1152$

(iv) $7! \times 2! = 10080$

(v) $1 \times 7! = 5040$

(vi) $2 \times 7 \times 3! \times 3! = 504$.

(vii) $2 \times 6!(5+4+3+2+1) = 21600$.

(b) (i) (α) $6! = 720$

(β) $\frac{4! \times 3!}{720} = \frac{144}{720} = \frac{1}{5}$

(γ) $\frac{4 \times 3! \times 3!}{720} = \frac{1}{5}$.

(ii) (α) $5! = 120$

(β) $\frac{3! \times 3!}{120} = \frac{36}{120} = \frac{3}{10}$

(γ) $\frac{2 \times 3!}{120} = \frac{1}{10}$.

(c) (i) $\frac{9!}{2!2!} = 90720$.

(ii) Total arrangements $= \frac{11!}{2!2!2!}$

$= 4989600$.

permutations 2M's together

$= \frac{10!}{2!2!}$

$= 907200$

$P(2M\text{'s}) = \frac{907200}{4989600} = \frac{2}{11}$.

(iii) $\{T, H, E\}$ # arrangement $= 3!$

permutations $= 3! \times \frac{9!}{2!2!}$

$= 544320$

$P(\text{THE together}) = \frac{544320}{4989600} = \frac{6}{55}$.

Question 2:

(a) (i) ${}^{8}C_3 = 56$

(ii) ${}^{4}C_2 \times {}^{4}C_1 = 6 \times 4 = 24$

(iii) 1 woman 2 men $= {}^{4}C_1 \times {}^{4}C_2 = 24$
2 women 1 man $= {}^{4}C_2 \times {}^{4}C_1 = 24$
3 women 0 men $= {}^{4}C_3 \times {}^{4}C_0 = 4$

Total $= 24 + 24 + 4 = 52$ ways.

(iv) Husband & wife cannot serve
$\Rightarrow$ 1 person from each couple is selected.

Selecting 3 couples from 4 couples
${}^{4}C_3 = 4$

Selecting a person from a couple
2 ways.

$\therefore$ total $= 4 \times 2 \times 2 \times 2$
$= 32$ ways.

(b) $p = 0.2 \quad q = 0.8 \quad n = 7$

(i) $P(4 \text{ correct}) = {}^{7}C_4 \, 0.2^4 \, 0.8^3$
$\doteqdot 0.029$ (3 dp)

(ii) P (at least one correct)
$= 1 - P$ (none correct)
$= 1 - 0.8^7$
$= 0.790$.

(iii) $\dfrac{T_{k+1}}{T_k} = \dfrac{n-k+1}{k} \cdot \dfrac{p}{q}$

$= \dfrac{8-k}{k} \times \dfrac{0.2}{0.8}$

$= \dfrac{8-k}{4k}$

Now $\dfrac{8-k}{4k} \geq 1$.

$8 - k \geq 4k$

$k \leq 1\frac{3}{5}$

$\therefore k = 1 \Rightarrow$ most likely number of correct answers is 1.

(c) let probability of a 6 on a single throw be p.

$${}^{10}C_3\, p^3(1-p)^7 = \frac{3}{4}\, {}^{10}C_1\, p(1-p)^9$$

$$120\, \frac{p^3}{p} = \frac{3}{4} \times 10\, \frac{(1-p)^9}{(1-p)^7}$$

$$120\, p^2 = \frac{15}{2}\, (1-p)^2$$

$$16p^2 = 1 - 2p + p^2$$

$$15p^2 + 2p - 1 = 0$$

$$(3p+1)(5p-1) = 0.$$

$$p = -1/3,\ 1/5.$$

but $p > 0 \Rightarrow p = \frac{1}{5}$

Question 3:

(a) general term : ${}^{20}C_r\, (x)^r \left(-\frac{1}{2x^3}\right)^{20-r}$

$$= {}^{20}C_r\ x^r\, (x^{-3})^{20-r} \left(-\frac{1}{2}\right)^{20-r}$$

term independent of $x \Rightarrow x^0$

$$\therefore\ r - 3(20-r) = 0$$
$$r - 60 + 3r = 0$$
$$4r = 60$$
$$r = 15.$$

$\therefore$ it is the 16th term :

$$T_{16} = {}^{20}C_{15}\, x^{15} \left(-\frac{1}{2x^3}\right)^5$$

$$= 15504 \times \frac{-1}{32}$$

$$= -\frac{969}{2}$$

(b) $\left(x+\frac{1}{x}\right)\left({}^9C_0\, x^9 + {}^9C_1\, x^8\left(-\frac{1}{x}\right) + {}^9C_2\, x^7\left(-\frac{1}{x}\right)^2 + {}^9C_3\, x^6\left(-\frac{1}{x}\right)^3 + \ldots + {}^9C_9\left(-\frac{1}{x}\right)^9\right)$

$= \left(x+\frac{1}{x}\right)\left(x^9 - {}^9C_1\, x^7 + {}^9C_2\, x^5 - {}^9C_3\, x^3 + {}^9C_4\, x - {}^9C_5\, \frac{1}{x} + {}^9C_7\, \frac{1}{x^3} - {}^9C_6\, \frac{1}{x^5} + {}^9C_7\, \frac{1}{x^7} - \frac{1}{x^9}\right)$

coefficient of x^2:

$${}^9C_4 - {}^9C_3 = 126 - 84 = 42$$

(c)

$$(1+x)^{2n} = 1 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + \dots + {}^{2n}C_n x^n + \dots + x^{2n} \quad (1)$$

$$(1+x)^n (1+x)^n = \left(1 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_{n-1} x^{n-1} + x^n\right) \times \left(1 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_{n-1} x^{n-1} + x^n\right) \quad (2)$$

The coefficient of x^n in (1) is ${}^{2n}C_n$

The coefficient of x^n in (2) is:

$${}^nC_0 \cdot {}^nC_n + {}^nC_1 \cdot {}^nC_{n-1} + {}^nC_2 \cdot {}^nC_{n-2} + \dots + {}^nC_r \cdot {}^nC_{n-r} + \dots + {}^nC_n \cdot {}^nC_0$$

but ${}^nC_r = {}^nC_{n-r}$

$$\Rightarrow ({}^nC_0)^2 + ({}^nC_1)^2 + \dots + ({}^nC_n)^2$$

$$\therefore {}^{2n}C_n = \sum_{k=0}^{n} ({}^nC_k)^2$$

(d) $(1-a)^4 = 1 - 4a + 6a^2 - 4a^3 + a^4$

$$a = x(2-x)$$

$$a^2 = x^2(2-x)^2 = x^2(4 - 4x + x^2) = 4x^2 - 4x^3 + x^4$$

$$a^3 = x^3(2-x)^3 = x^3(8 - 3(2)^2 x + 3(2)x^2 - x^3) = x^3(8 - 12x + 6x^2 - x^3) = 8x^3 - 12x^4 + 6x^5 - x^6.$$

$(1-a)^4$

$$= 1 - 4x(2-x)) + 6(4x^2 - 4x^3 + x^4) - 4(8x^3 - 12x^4 + 6x^5 - x^6) + \dots$$

$$= 1 - 8x + 4x^2 + 24x^2 - 24x^3 + 6x^4 - 32x^3 + 48x^4 - 24x^5 + 4x^6 + \dots$$

$$= 1 - 8x + 28x^2 - 56x^3 + 54x^4 + \dots$$

$\therefore$ $T1 = 1$

$T2 = -8x$

$T3 = 28x^2$

$T4 = -56x^3.$

(e) $(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$

Integrating both sides w.r.t. x

$$\frac{1}{n+1}(1+x)^{n+1} = {}^nC_0 \frac{x}{1} + {}^nC_1 \frac{x^2}{2} + {}^nC_2 \frac{x^3}{3} + \dots + {}^nC_n \frac{x^{n+1}}{n+1} + K.$$

When $x = 0$

$$\frac{1}{n+1} = 0 + K.$$

$$\therefore K = \frac{1}{n+1}.$$

$$\Rightarrow \frac{1}{n+1}(1+x)^{n+1} - \frac{1}{n+1} = {}^nC_0 \frac{x}{1} + {}^nC_1 \frac{x^2}{2} + \dots + {}^nC_n \frac{x^{r}}{n+1}$$

When $x = 1$.

$$\frac{2^{n+1} - 1}{n+1} = \frac{{}^nC_0}{1} + \frac{{}^nC_1}{2} + \frac{{}^nC_2}{3} + \dots \frac{{}^nC_n}{n+1}.$$

$$\therefore \frac{1}{n+1}\left(2^{n+1} - 1\right) = \sum_{r=0}^{n} \frac{{}^nC_r}{r+1}$$