APPLICATIONS OF CALCULUS TO THE PHYSICAL WORLD

1) The rate of change in the number of koalas in a certain area is given by $R = 54 - 2t$ where t is time in months. If there are initially 640 koalas in the area, find
(a) how many koalas there are after a year
(b) when there are no koalas left.

2) The surface area of a disc in mm^2 expands as it heats up according to the formula $A = 5 + t^2$ where t is time in seconds. Find the rate at which the surface area is expanding after 5 seconds.

3) Point P moves along the curve $y = x^3 - 2x^2 + 1$. Find the rate of change in the y-coordinate of P if the x-coordinate is increasing at the rate of 2 units per second when $x = 5$.

4) The radius of a spherical ball is increasing at a constant rate of 0.8 cms^{-1} as the ball expands when heat is applied. Find the rate of increase in the volume of the ball when the radius is 10 cm.

5) The volume of an icecube is decreasing at a constant rate of 0.2 mm^3s^{-1} as the icecube melts. At what rate is the side of the icecube decreasing when it has side 20 mm?

6) The volume of a cube is increasing at a constant rate of 3 mm^3s^{-1}. Find the rate of increase in its surface area when its side is 50 mm.

7) The mass of a chemical is given by $M = 100e^{0.023t}$ where M is its mass in grams and t is time in hours.
(a) What is the initial mass?
(b) What is the growth rate as a percentage?
(c) What is the mass after 7 hours, correct to one decimal place?
(d) What is the rate of increase in the mass after 7 hours, to two significant figures?
(e) Find the time, to the nearest hour, taken for the chemical to reach a mass of 500 g.

8) The population of a certain city is given by the formula $P = Ae^{kt}$ where t is the number of years. Initially the population is 55 000, and after 5 years it has increased to 68 000.
(a) Find the values of A and k. (Give k to three significant figures)
(b) Find the population after 10 years.
(c) Find the rate of population growth after 10 years.
(d) Find the time it takes for the population to increase to 1 000 000, to the nearest year.

9) A mass of uranium decays from 300 g to 260 g in 7 years. Find, to one decimal place,
(a) its half life (the time it takes to reach half its initial mass)
(b) how long it takes to decay to 50 g.

10) In summer, the number of red algae in a river satisfies the equation $N(t) = Ae^{0.12t}$ where t is measured in days.
(a) Show that the number of algae increases at a rate proportional to the number

present.
(b) Find how long it takes for the number of algae to increase by 20% (to one decimal place).
(c) After 5 days, the number of algae is estimated as 2.3×10^{11}. Evaluate A to two significant figures.
(d) The number of algae doubles every n days. Find the value of n to one decimal place.

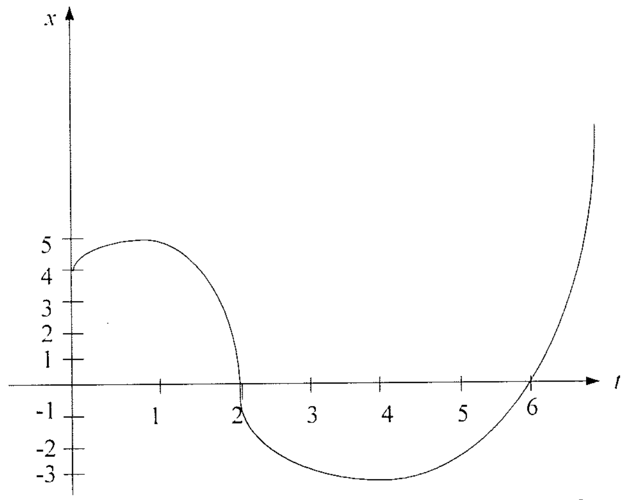
11) The population of a certain bird colony is given by $P(t) = P_0 e^{kt}$. If the population decreases by 15% after 2 years, find
(a) the value of k to three significant figures.
(b) when the population is decreased by 40%. Answer to one decimal place.

12) The growth rate in a certain population of birds is given by $\frac{dP}{dt} = k(P - 3000)$.
(a) Show that $P = 3000 + Ae^{kt}$ is a solution of the above equation.
(b) If the bird population is initially 15390 and after 2 years it is 17420, find the values of A and k.
(c) What is the population after 10 years?
(d) When would the population reach 1 000 000?

13) A piece of metal is heated to 95^0C and placed in a room with a constant air temperature of 25^0C.
(a) If the metal cools to 90^0C after 5 minutes, what will its temperature be after 15 minutes?
(b) When will the temperature reach 30^0C?

14) The volume of water in a dam is given by $V = 800 + 300e^{kt}$ where k is a constant. The volume increases by 5% after 6 hours.
(a) How long would it take for the volume to increase by 20%?
(b) Find the percentage increase in volume after 24 hours.

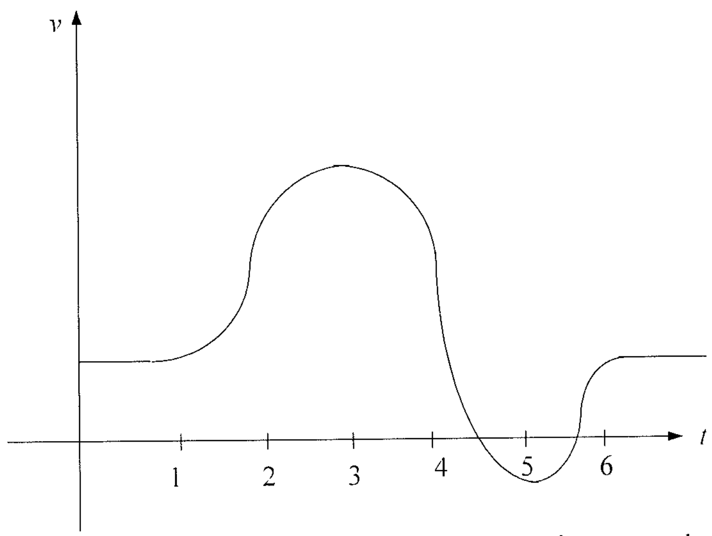
15)



The graph shows the displacement x metres of a particle at time t seconds.
(a) When is the particle at rest?
(b) When is the particle at the origin?

(c) How far does the particle travel in the first 6 seconds?
(d) Draw a rough sketch showing the velocity of the particle.

16)



The velocity of a particle is shown above as it moves along a straight line .
(a) At what times is the acceleration of the particle zero?
(b) Between what times is the acceleration at its greatest?
(c) At what times is the particle at rest?

17) A car starts and drives off, increasing its speed at a constant rate until it reaches 30 m/s after 10 seconds. It travels at this speed for 5 seconds, then it slows down at a constant speed for 5 seconds until it is travelling at 20 m/s. It travels at this speed for 15 seconds.
(a) Sketch the speed v as a function of time t.
(b) Graph the distance travelled by the car over time t on a separate diagram.

18) The displacement of a particle in cm is given by $x = 2t^3 - 21t^2 + 60t$. at time t seconds. Find
(a) the displacement after 3 seconds
(b) the initial velocity
(c) the acceleration after 5 seconds
(d) the times when the particle is at rest
(e) the total displacement over the first 3 seconds.

19) The height of a ball in metres over time t seconds is given by $h = 1 + 15t - 5t^2$.
(a) Find the maximum height reached.
(b) Show that the acceleration is constant.
(c) Find the time, to one decimal place, that the ball reaches the ground.

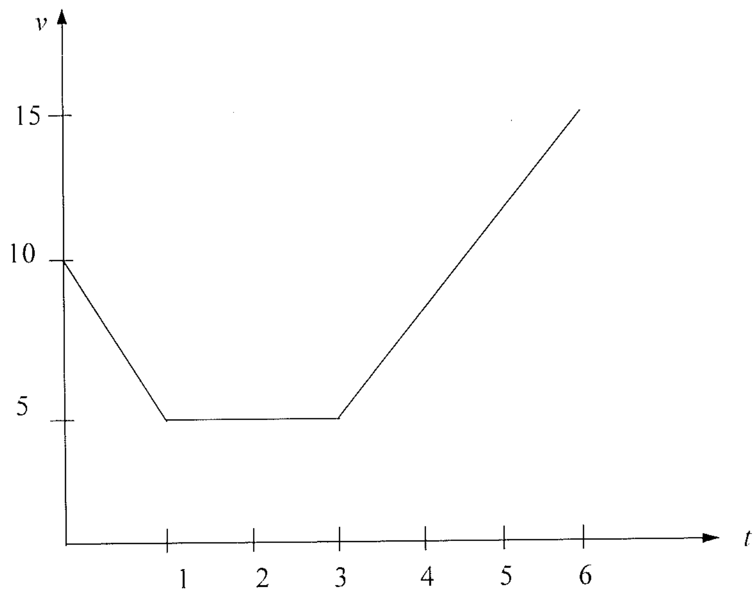
20) The displacement of a spring is given by $s = \cos 2t$ cm at time t seconds. Find
(a) the initial velocity and acceleration
(b) the times when the spring is at rest
(c) the displacement of the spring at these times.

(d) Show that the acceleration of the spring is given by $\ddot{s} = -4s$

21) A particle moves in a straight line so that its displacement is given by $x = 2e^{3t}$ cm at time t seconds. Find
(a) the exact value of the velocity after 9 seconds
(b) the exact time when the acceleration is 36 cms^{-2} .

22) A particle moves along a straight line and its displacement from a fixed point O is given by $x = 5 - 3t + 7\ln(t + 1)$ metres, where t is time measured in seconds.
(a) Find the initial position of the particle.
(b) Find the velocity of the particle after 3 seconds.
(c) Find the acceleration of the particle at time t seconds.

23)



The graph above shows the velocity of a particle in cm/second. Find the total displacement of the particle in the first 6 seconds.

24) The velocity of a particle is $v = 3t^2 - 12t$ ms^{-1} . If the initial displacement is 3 m, find the displacement of the particle after 5 seconds.

25) The acceleration of a particle is given by $a = 18e^{3t}$ cms^{-2}. If the particle is at rest at the origin after 1 second, find the exact displacement of the particle after 3 seconds.

26) Two particles M and N start moving along the x-axis at time $t = 0$. Particle M is initially at $x = 4$ and its velocity at time t is given by $v = 4t - 6$. Particle N has its displacement given by $x = 2t - 2$.
(a) Find the equation of displacement for particle M.
(b) When do the two particles meet?

27) The velocity of a particle after t seconds is $v = 3\cos 3t$ ms^{-1}. The particle is initially 2 m to the left of the origin. Find
(a) the acceleration of the particle after $\frac{\pi}{6}$ seconds
(b) the displacement of the particle after $\frac{\pi}{6}$ seconds.

28) The velocity of a particle is given by $v = \frac{1}{6e^{3x}}$ ms^{-1} where x is the displacement of the particle. If the particle is initially at the origin, find the equation of the displacement of the particle in terms of t.

29) The acceleration of a particle is given by a = $12x - 6$ ms^{-2}. The particle has an initial velocity of 3 ms^{-1} and is 1 m to the right of the origin. Find the equation for the velocity of the particle.

30) The acceleration of a particle is given by $\ddot{x} = 6x$ ms^{-2}. The particle is travelling at 2 ms^{-1} when it is at the origin. Find the velocity of the particle when it is 5 m to the right of the origin.

31) A pendulum moves so that its displacement in cm over time t seconds is given by $x = \cos 2t$.

(a) Show that its acceleration is $\ddot{x} = -4x$.
(b) Find the times at which the pendulum is at the origin.
(c) Find the velocity when the pendulum is at the origin.
(d) Find its maximum acceleration.

32) A spring moves in simple harmonic motion with acceleration $\frac{d^2x}{dt^2} = -9x$ mms^{-2}.

(a) Show that $x = 5 \cos 3t$ is a formula for the displacement of the spring.
(b) Find the amplitude and period of the motion.
(c) What is the maximum speed of the spring?
(d) What is its acceleration when the spring is at the origin?

33) The velocity of a particle moving in SHM is given by $v^2 = 5x - x^2$ cms^{-1}.
(a) Find the two points between which the particle is oscillating.
(b) Find the centre of the motion.
(c) Find the maximum speed of the particle.
(d) Find its acceleration in terms of its displacement x.

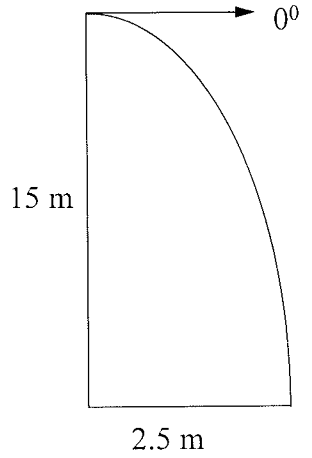
34) A particle is moving in SHM with amplitude 2 cm and period 4 seconds. Find its exact velocity and acceleration when its displacement is 1 cm to the right of the centre of motion.

35) The displacement of a particle is given by $s = 4 \sin 2t + 3 \cos 2t$ cm.
(a) Show that the particle is moving in simple harmonic motion.
(b) Find the amplitude of the particle.
(c) Find its maximum speed.

36) A ball is thrown at 8 ms^{-1} at an angle of 50^0.
(a) Find the ball's maximum height reached.
(b) Find the time taken for the ball to land (to one decimal place).
(c) How far away will the ball land from where it was thrown?
Use 10 ms^{-2} as the acceleration due to gravity and neglect air resistance.

37) Anne fires an arrow at a speed of 25 ms^{-1} and aims at the centre of a target 1.2 m high and 40 metres away. At what angle should Anne fire the arrow so that it hits the centre of the target?
Use 10 ms^{-2} as the acceleration due to gravity and neglect air resistance.

38) Lee drops a book out of a window 15 metres high. He aims at a point on the ground 2.5 metres out from the foot of the building.



What would the initial velocity of the book have to be to hit the ground at that point? Use 9.8 ms^{-2} as the acceleration due to gravity and neglect air resistance.

ANSWERS

1) (a) 1 144 (b) 64 months
2) 10 mm^2/s
3) 110 units per second
4) 1005.3 cm^3s^{-1}
5) 1.7×10^{-4} mms^{-1}
6) 0.24 mm^2s^{-1}
7) (a) 100 g (b) 2.3% (c) 117.5 g (d) 2.7 g/hour (e) 70 hours
8) (a) A = 55 000, $k = 0.0424$ (b) 84 073 (c) 3568 people/year (d) 68 years
9) (a) 33.9 years (b) 87.6 years
10) (a) $N(t) = Ae^{0.12t}$

$N^1(t) = 0.12Ae^{0.12t}$

$= 0.12N(t)$

So rate is proportional to the number of algae.

(b) 1.5 days (c) $A = 1.3 \times 10^{11}$ (d) n = 5.8

11) (a) $k = -0.0813$ (b) 6.3 years
12) (a) $P = 3000 + Ae^{kt}$

$\therefore Ae^{kt} = P - 3000$

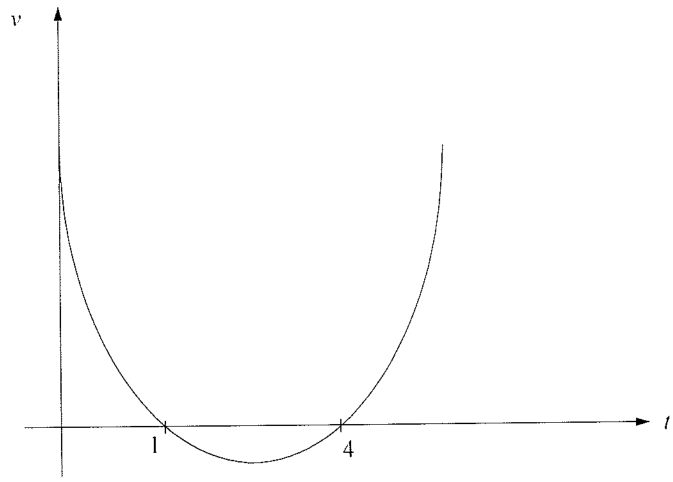
$$\frac{dP}{dt} = kAe^{kt}$$

$$= k(P - 3000)$$

(b) $A = 12\,390$, $k = 0.07586$ (c) 29 457 birds (d) 57.8 years

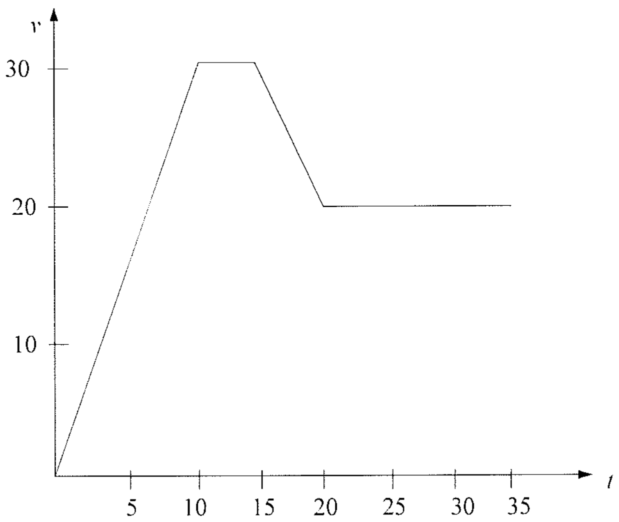
13) (a) 81^0C (b) 178 minutes (2 hours and 58 minutes)
14) (a) 19.6 hours (b) 26.2%
15) (a) $t = 1, 4$ s (b) $t = 2, 6$ s (c) 12 metres

(d)

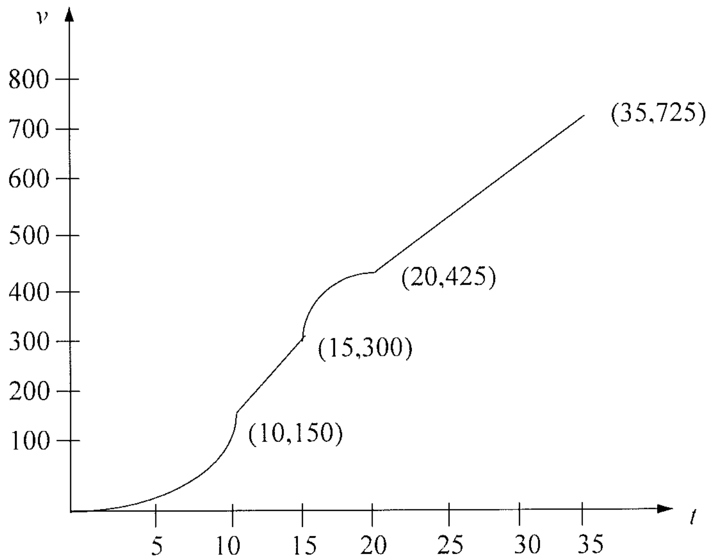


16) (a) $t = 3, 5$ (b) $3 < t < 5$ (c) $0 \le t \le 1, t \ge 6$

17) (a)



(b)



18) (a) 45 cm (b) 60 cms^{-1} (c) 18 cms^{-2} (d) t = 2, 5 s (e) 59 cm

19) (a) 12.25 m (b) a = -10 ms^{-2} (c) After 3.1 s

20) (a) $v = 0$ cms^{-1} , a = -4 cms^{-2} (b) $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$ s (c) ± 1 cm

(d) $s = \cos 2t$

$\dot{s} = -2\sin 2t$

$\ddot{s} = -4\cos 2t$

$= -4s$

21) (a) $6e^{27}$ cms^{-1} (b) $\frac{\log_e 2}{3}$ s

22) (a) 5 m (b) -1.25 ms^{-1} (c) $-\frac{7}{(t+1)^2}$ ms^{-2}

23) 47.5 cm

24) -22 m

25) $2e^3(e^6 - 7)$ cm

26) (a) $x = 2t^2 - 6t + 4$ (b) $t = 1, 3$ s

27) (a) -9 ms^{-2} (b) -1 m

28) $x = \frac{\ln\left(\frac{t}{2}+1\right)}{3}$

29) $v = \sqrt{12x^2 - 12x + 9}$

30) 12.4 ms^{-1}

31) (a) $x = \cos 2t$

$$\dot{x} = -2\sin 2t$$

$$\ddot{x} = -4\cos 2t$$

$$= -4x$$

(b) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$ seconds (c) ± 2 cms^{-1} (d) 4 cms^{-2}

32) (a) $x = 5\cos 3t$

$$\frac{dx}{dt} = -15\sin 3t$$

$$\frac{d^2x}{dt^2} = -45\cos 3t$$

$$= -9(5\cos 3t)$$

$$= -9x$$

(b) Amplitude 5, period $\frac{2\pi}{3}$ (c) 15 mms^{-1} (d) 0 mms^{-2}

33) (a) 0, 5 cm (b) 2.5 cm (c) 6.25 cms^{-1} (d) $a = 25x - 15x^2 + 2x^3$

34) $-\frac{\sqrt{3}\pi}{2} cms^{-1}; -\frac{\pi^2}{4} cms^{-2}$

35) (a) $s = 4\sin 2t + 3\cos 2t$

$$\frac{ds}{dt} = 8\cos 2t - 6\sin 2t$$

$$\frac{d^2s}{dt^2} = -16\sin 2t - 12\cos 2t$$

$$= -4(4\sin 2t + 3\cos 2t)$$

$$= -4s$$

So the particle is in SHM

(b) 5 (c) 10 cms^{-1}

36) (a) 1.9 m (b) 1.2 s (c) 6.3 m
37) $21^0 53'$, $69^0 50'$
38) 1.43 ms^{-1}