Brigidine College Randwick

Name :

Year 12 Extension 1 Mathematics
June 2004 Assessment Task on Application of Calculus

Time Allowed : 45mins
Show all necessary working

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1. If $T = P + Ae^{-kt}$ and P,A and k are constants, show that $\frac{dT}{dt} = -k(T - P)$ **(2)**

2. A particle is moving so its displacement is given by $x = 3\text{Sin}4t$. Show that it is moving in simple harmonic motion. **(2)**

3. A hollow vessel in the shape of a parabola $16x^2 = 9y$ is rotated about its' y-axis of symmetry. The width of the parabola at the open end is 1.5 m. Water is poured into the vessel at a constant rate of $\frac{3\pi}{8}\text{ m}^3$ per minute. **(8)**

   a. Show that when the depth of the water in the vessel is h metres, the volume of water is $\frac{9\pi h^2}{32}\text{ m}^3$.

   b. Show that the depth of the water when it is half full is $\frac{\sqrt{2}}{2}$.

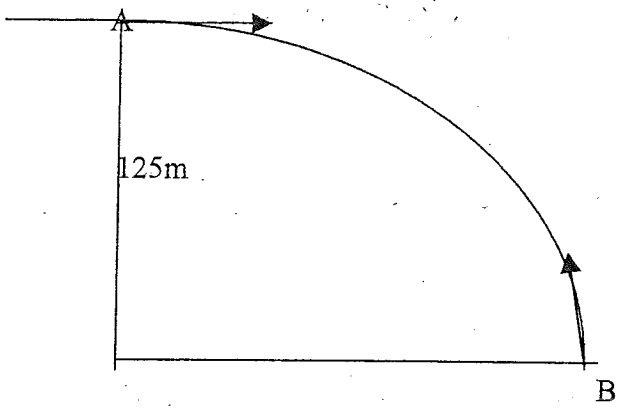
   c. At what rate is the depth of the water increasing at the instant when it is half full.

4. The motion of a particle moving on the x-axis is governed by the equation $\frac{d^2x}{dt^2} = 2e^x - 3e^{-x}$ where x metres is the displacement from the origin O after t seconds. **(6)**

   a. Prove $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}(v^2)\right)$.

   b. If $\frac{dx}{dt} = 0$ when $x = 0$, show that $\left(\frac{dx}{dt}\right)^2 = 4e^x + 6e^{-x} - 10$ and hence or otherwise, find the other value of x for which $\frac{dx}{dt} = 0$.

5. A particle is fired horizontally with a velocity of 40m/s from a point A on the top of a vertical cliff so that it hits a boat B. A is 125 m above sea level **(6)**



Assuming there is no air resistance and taking acceleration due to gravity to be $10m/s^2$, find

a. the time taken for the particle to reach the boat.

b. the horizontal distance from A to B.

c. If the boat fires the particle back so that it passes through point A horizontally and assuming that the particles takes the same amount of time to travel as in "part a", determine

i. the angle of projection from B.

ii. the velocity of projection from B.

6. A 100L tank is filled with cordial containing 60kg of dissolved sugar. Water runs into the tank at a rate of 2L/min , (the mixture is kept uniform by stirring) and runs out at the same rate. **(4)**

a. If Q = the amount of sugar in kg in the tank after t minutes , show that $\frac{dQ}{dt} = -0.02Q$.

b. How much sugar is left in the tank after 2 hours?

7. A particle is initially at rest and 4 metres from the centre of motion exhibits simple harmonic motion. If the particles' speed is 12m/s when it passes the equilibrium point, find the particles' **(4)**

a. period after 2 seconds.

b. acceleration after 2 seconds.

Q1/ $T = P + Ae^{-kt}$

$\frac{dT}{dt} = -kAe^{-kt}$ ✓

but from $T = P + Ae^{-kt}$

$Ae^{-kt} = T - P$ → sub into $\frac{dT}{dt}$ ✓

$\therefore \frac{dT}{dt} = -k(T-P)$

Q2/ $x = 3\sin 4t$

$\dot{x} = 12\cos 4t$ ✓

$\ddot{x} = -48\sin 4t$

$= -16 \times 3\sin 4t$

$= -16x = -4^2 x$ ✓

which is in the form $\ddot{x} = -n^2 x$

Q4/ a) $\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$ ✓

$= \frac{dv}{dx} \times v$ ✓

$= \frac{dv}{dx} \times \frac{d}{dv}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ ✓

b) $\frac{d^2x}{dt^2} = 2e^x - 3e^{-x}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2e^x - 3e^{-x}$

$\frac{1}{2}v^2 = \int 2e^x - 3e^{-x}\,dx$

$\frac{1}{2}v^2 = 2e^x + 3e^{-x} + C$

$v = 0$ when $x = 0$

$0 = 2 + 3 + C$

$\therefore C = -5$ ✓

$\frac{1}{2}v^2 = 2e^x + 3e^{-x} - 5$

$v^2 = 4e^x + 6e^{-x} - 10$ ✓

when $v = 0$, $x = ?$

$0 = 4e^x + 6e^{-x} - 10$

$0 = 4e^{2x} + 6 - 10e^x$

$0 = 4e^{2x} - 10e^x + 6$

$0 = 2e^{2x} - 5e^x + 3$

let $e^x = M$

$0 = 2M^2 - 5M + 3$

$2M \times -3$

$M \times -1$

$0 = (2M-3)(M-1)$ ✓

$M = \frac{3}{2}$ $M = 1$

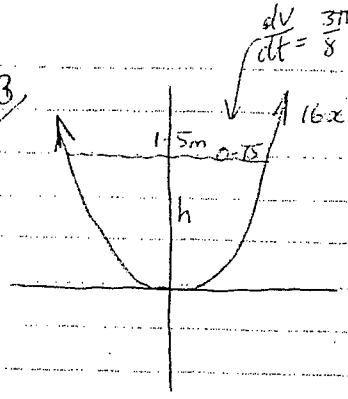
$e^x = \frac{3}{2}$ $e^x = 1$

$\ln e^x = \ln \frac{3}{2}$ $x = 0$

$x = \ln \frac{3}{2}$ ✓

or 0.41

Q3,



a) $V = \pi \int_0^h x^2\,dy$

$V = \pi \int_0^h \frac{9y}{16}\,dy$

$V = \pi \left[\frac{9y^2}{32}\right]_0^h$

$V = \pi \left[\frac{9h^2}{32} - 0\right]$ ✓✓

$V = \frac{9\pi h^2}{32}$

b) when full $x = 0.75$ $y = ?$

$16 \times 0.75^2 = 9y$ ✓

$y = 1$

$\therefore V = \frac{9\pi \times 1^2}{32}$ ✓

$V = \frac{9\pi}{32}$

$\therefore \frac{1}{2}$ full $V = \frac{1}{2} \times \frac{9\pi}{32} = \frac{9\pi}{64}$

$\therefore$ depth, ie $h = ?$ when $V = \frac{9\pi}{64}$

$\frac{9\pi}{64} = \frac{9\pi h^2}{32}$ ✓

$h^2 = \frac{1}{2}$

$h = \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$ or 0.71

c) $\frac{dV}{dt} = \frac{3\pi}{8}$, need $\frac{dV}{dh}$ first ✓✓

$\frac{dh}{dt} = ?$, when $h = \frac{1}{\sqrt{2}}$

$V = \frac{9\pi h^2}{32}$

$\frac{dV}{dh} = \frac{18\pi h}{32}$

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$

$= \frac{3\pi}{8} \times \frac{32}{18\pi h}$, when $h = \frac{1}{\sqrt{2}}$

$\frac{dh}{dt} = \frac{3\pi}{8} \times \frac{32}{18\pi} \times \frac{1}{\sqrt{2}}$

$\frac{dh}{dt} = \frac{2\sqrt{2}}{9}$ ✓

#5

a) $\ddot{x} = 0$ $\quad \ddot{y} = -10$

$\dot{x} = V\cos\alpha$ $\quad \dot{y} = -10t + C$

$\alpha = 0$ $\quad t = 0$ $\dot{y} = V\sin\alpha$ but $\alpha = 0$

$\therefore \dot{x} = V$ $\quad \therefore \dot{y} = V\sin 0$

$\dot{y} = 0$

$\therefore C = 0$

$\dot{x} = V$ $\quad \dot{y} = -10t$

$x = Vt$ $\quad y = -\frac{10t^2}{2} + C$

$t = 0$, $y = 125$ $\therefore C = 125$

$y = -\frac{10t^2}{2} + 125$ ✓✓

when $y = 0$, $t = ?$

$0 = -\frac{10t^2}{2} + 125$

$t = 5$

b) $x = 40 \times 5$ ✓

$x = 200$ m

c) i) $\ddot{x} = 0$ $\quad \ddot{y} = -10$

$\dot{x} = V\cos\alpha$ $\quad \dot{y} = -10t + C$

$t = 0$ $\dot{y} = V\sin\alpha$

$V\sin\alpha = C$

$\therefore \dot{y} = -10t + V\sin\alpha$

$x = Vt\cos\alpha$ $\quad y = -\frac{10t^2}{2} + Vt\sin\alpha + C$

$t = 0$, $y = 0$ $\therefore C = 0$

$y = -\frac{10t^2}{2} + Vt\sin\alpha$

~~when $y = 125$~~

when $t = 5$, $x = 200$, $y = 125$, $\alpha = ?$

$200 = 5V\cos\alpha \cdots ①$

also $125 = -125 + 5V\sin\alpha$

$250 = 5V\sin\alpha \cdots ②$

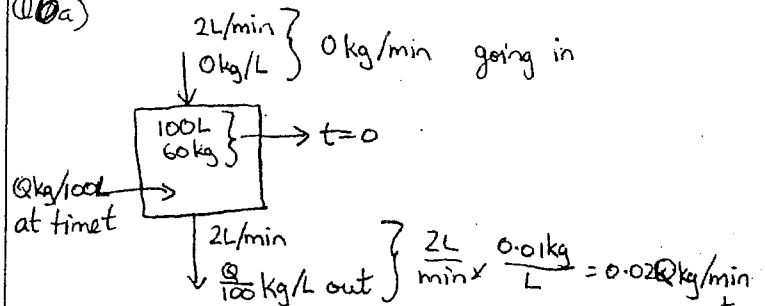
② ÷ ① $\tan\alpha = 1.25$ ✓

$\alpha = 51° 20'$

ii) $200 = 5V\cos 51°20'$

$\therefore V = 64$ m/s ✓

Q6a)



$$\frac{dQ}{dt} = \text{in} - \text{out}$$

$$\frac{dQ}{dt} = 0 - 0.02Q$$

$$\frac{dQ}{dt} = -0.02Q \quad \checkmark$$

b) $$\frac{dt}{dQ} = \frac{1}{-0.02Q}$$

$$t = \int \frac{1}{-0.02Q}\,dQ = -\frac{1}{0.02}\int \frac{1}{Q}\,dQ = -50\int \frac{1}{Q}\,dQ$$

$$t = -50\ln Q + c$$

$t = -50\ln Q + c$

$t=0 \quad Q=60 \quad \therefore c = 50\ln 60$ ✓

$t = -50\ln Q + 50\ln 60$

$0.02t = \ln 60 - \ln Q$

$0.02t = \ln \frac{60}{Q}$

$\frac{60}{Q} = e^{0.02t}$

$Q = \frac{60}{e^{0.02t}} = 60e^{-0.02t}$ ✓

when $t = 2$ hrs (120 mins)

$Q = 60e^{-0.02\times 120} = 5.44$ kg ✓

Q7a) $x = a\cos(nt+\alpha)$ → starts off origin

$\dot{x} = -an\sin(nt+\alpha)$

$\ddot{x} = -an^2\cos(nt+\alpha)$ ✓

$t=0 \quad \dot{x}=0$

$0 = -an\sin(\alpha)$

$\sin\alpha = 0$

$\alpha = 0$

$t=0 \quad x=4$

$4 = a\cos 0$

$a = 4$ ✓

$V^2 = n^2(a^2 - x^2)$

$V=12, \quad x=0$

$144 = n^2(4^2 - 0^2)$

$n^2 = 9$

$n = 3$ → period $= \frac{2\pi}{3}$ ✓

b) acceleration at $t=2$

$\ddot{x} = -4(3)^2\cos(3\times 2)$ ✓

$\ddot{x} = -34.57$