J.M.J.

MARCELLIN COLLEGE RANDWICK

YEAR 12 HSC

ASSESSMENT TASK # 1

EXTENSION I MATHEMATICS

2007

Weighting: 30% of H.S.C. Assessment Mark.

STUDENT NAME: ______________	MARK:		/46
	PERCENTAGE:		%
	RANK ON THIS TASK:		/23

Time Allowed: $1\frac{1}{2}$ hours

Directions:
- Answer all questions on separate lined paper.
- Show all necessary working.
- Marks may not be awarded for careless or badly arranged work.

Outcomes examined:
PE2 – Uses multi-step deductive reasoning in a variety of contexts.
PE3 – Solves problems involving polynomials and parametric representations.
PE4 – Uses the parametric representation together with differentiation to identify geometric properties of parabolas.
PE6 – Makes comprehensive use of Mathematical language, diagrams and notation for communicating in a wide variety of situations.
HE1 – Appreciates interrelationships between ideas drawn from different areas of Mathematics.
HE2 – Uses inductive reasoning in the construction of proofs.
HE3 – Uses a variety of strategies to investigate Mathematical models of situations involving projectile motion, simple harmonic motion, or exponential growth and decay.
HE5 – Applies the chain rule to problems including those involving velocity and acceleration as functions of displacement.

QUESTION ONE (2 MARKS)

Marks

Consider the polynomial $P(x) = 2x^3 + 3x^2 - kx + 12$

(a) Determine the value of k if $x + 4$ is a factor of $P(x)$ 1

(b) Hence express $P(x)$ as a product of its linear factors 1

QUESTION TWO (3 MARKS)

A plane flying horizontally at 500 km/h releases a projectile designed to hit a target on the ground. The plane is flying at a constant height of 2 km. 3

You may assume the displacement-time equations of motion:

$x = Vt\cos\theta$ and $y = \dfrac{-gt^2}{2} + Vt\sin\theta + 2000$

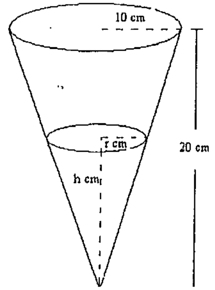
and that $g = 10ms^{-2}$

Calculate the horizontal distance from the target that the plane must release the projectile to successfully hit the target.

QUESTION THREE (4 MARKS)

Water is running out of a conical funnel at the rate of $5cm^3/s$. The base radius of the funnel is 10cm and the height is 20cm.

Let h cm be the height and r cm be the base radius of the remaining water.



NOT TO SCALE

(a) Show that $r = \frac{1}{2}h$, giving reason(s). 1

(b) Show that the volume (V) of water in the cone can be expressed as: 1

$$V = \frac{1}{12}\pi h^3$$

(c) How fast is the water level dropping when the water is 10cm deep? 2

QUESTION FOUR (4 MARKS)

Marks

The velocity of a particle in terms of its displacement is given by $v = \sqrt{3x+1}$ where x is the displacement in metres and v is the velocity in metres per second. The particle is initially at the origin.

(a) Show that the acceleration of the particle is a constant. 1

(b) Find its displacement after 5 seconds 3

QUESTION FIVE (4 MARKS)

(a) Show that $\sqrt{3}\cos 2t - \sin 2t = 2\cos(2t + \frac{\pi}{6})$ 1

(b) A particle moves in a straight line and its displacement x metres at any time t seconds is given by:

$$x = 5 + \sqrt{3}\cos 2t - \sin 2t$$

i) Prove that the particle's motion is Simple Harmonic 2

ii) Between what two points is the particle oscillating? 1

QUESTION SIX (3 MARKS)

The temperature of a particular body satisfies an equation of the form $T = B + A_e^{-kt}$ where T is the temperature of the drink in degrees Celsius, t is the time in minutes, A and k are constants and B is the temperature of the surroundings in degrees Celsius.

The body cools from 90°C to 80°C in 2 minutes in a surrounding of temperature 30°C.

(a) Find the values of A and k 2

(b) Find the temperature of the body after a further 5 minutes have passed (Answer correct to the nearest degree) 1

QUESTION SEVEN (6 MARKS)

Marks

A particle is undergoing Simple Harmonic Motion, oscillating between the points P at x = 3 and Q at x = -5 on the x axis. It takes $\frac{\pi}{2}$ seconds for the particle to travel from P to Q.

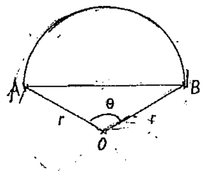
(a) Write down its acceleration in terms of x 2

(b) Find its maximum acceleration 1

(c) Find its maximum speed 3

QUESTION EIGHT (4 MARKS)

Consider a sector of a circle of radius r, the angle at the centre being θ



(a) Show that when $\sin\theta = \frac{\theta}{2}$ the chord AB bisects the sector 1

(b) Investigate whether 1.8 or 2.0 would be a more satisfactory first approximation for the solution of the equation $\sin\theta - \frac{\theta}{2} = 0$. 1

(c) Use Newton's method once to obtain a better approximation of the root. (Use your answer from (b) as an initial approximation). Answer correct to 2 decimal places 2

QUESTION NINE (4 MARKS)

P($2ap$, ap^2) is any point on the parabola $x^2 = 4ay$. The line k is parallel to the tangent at P and passes through the focus, S, of the parabola.

(a) Find the equation of the line k 1

(b) The line k intersects the x-axis at the point Q. Find the coordinates of the midpoint, M, of the interval QS. 2

(c) What is the equation of the locus of M? 1

QUESTION TEN (5 MARKS)

Marks

(a) Prove that:

i) $-2\cos\left(x+\frac{\pi}{6}\right) = \sin x - \sqrt{3}\cos x$ 1

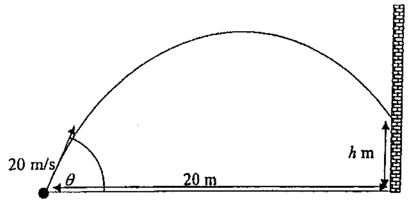
ii) $\tan^2 x - 3 = \dfrac{\sin^2 x - 3\cos^2 x}{\cos^2 x}$ 1

(b) Hence evaluate 3

$$\lim_{x \to \frac{\pi}{3}} \frac{\tan^3 x - 3\tan x}{\cos\left(x+\frac{\pi}{6}\right)}$$

QUESTION ELEVEN (7 MARKS)

A ball is fired from level ground at 20m/s, aiming to hit as high as it can up a wall 20m away (In this problem, take g = 10m/s^2)



(a) Prove that, for any point P(x,y) on the ball's path 2

$$x = 20\,t\cos\theta \text{ and } y = 20t\sin\theta - 5t^2$$

(b) Prove that the height h on the wall obtained by firing the ball at an angle θ is given by 1

$$h = 20\tan\theta - 5\sec^2\theta$$

(c) Prove that 1

$$\frac{dh}{d\theta} = 10\sec^2\theta(2 - \tan\theta)$$

(d) Find the maximum height the ball can reach up the wall 3

2007 HALF - YEARLY EXAM

ASSESSMENT TASK 1

EXTENSION 1 MATHS

Question One

a) If $x+4$ is a factor of $P(x)$ then $P(-4)=0$

$\therefore -128 + 48 + 4k + 12 = 0$

$\therefore k = 17$ ← 1 mark

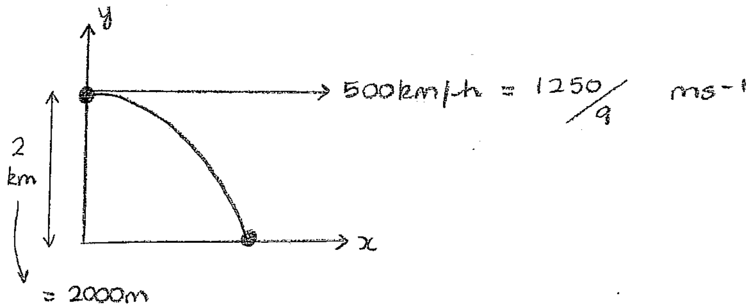
b)
$$\begin{array}{r|l} & 2x^2 - 5x + 3 \\ \hline x+4 & 2x^3 + 3x^2 - 17x + 12 \\ & 2x^3 + 8x^2 \\ \hline & -5x^2 - 17x + 12 \\ & -5x^2 - 20x \\ \hline & 3x + 12 \\ & 3x + 12 \\ \hline & 0 \end{array}$$

$\therefore P(x) = (x+4)(2x^2-5x+3)$

$= (x+4)(2x+1)(x-3)$ ← 1 mark

$2x-3$

Question Two



$$y = -\frac{10t^2}{2} + \frac{1250t\sin 0}{9} + 2000$$

$$\therefore y = -5t^2 + 2000 \quad \longleftarrow \text{1 mark}$$

Now, projectile hits the ground when $y = 0$

$$\therefore 5t^2 = 2000$$

$$\therefore t = \pm 20$$

$$\therefore t = 20 \text{ secs} \quad (\text{since } t \geq 0) \quad \longleftarrow \text{1 mark}$$

when $t = 20$ $\quad x = \frac{1250}{9}(20)\cos 0°$

$$\therefore x = \frac{25000}{9} \text{ m} \quad \longleftarrow \text{1 mark}$$

Question Three

a) Using similar Δ's:

$$\frac{r}{10} = \frac{h}{20}$$

$$\therefore r = \frac{h}{2}$$

1 mark

b) $V = \frac{1}{3}\pi r^2 h$

$$\therefore V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$\therefore V = \frac{\pi h^3}{12}$$

1 mark

c) $\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$ $\qquad \frac{dV}{dh} = \frac{\pi h^2}{4}$

$$= \frac{4}{\pi h^2} \cdot (-5)$$

1 mark

$$= -\frac{20}{\pi h^2}$$

when $h = 10$, $\frac{dh}{dt} = -\frac{1}{5\pi}$ $\quad \longleftarrow$ 1 mark

$\therefore$ The water level is dropping at $\frac{1}{5\pi}$ cms^{-1}

Question Four

a) $v = \sqrt{3x+1}$

Now $v^2 = 3x+1$

$\therefore \frac{v^2}{2} = \frac{3x}{2} + \frac{1}{2}$

$\therefore \frac{d}{dx}\left(\frac{v^2}{2}\right) = \frac{3}{2}$

$\therefore \ddot{x} = 3/2$

$\therefore$ Acc. is constant

1 mark

b) $\frac{dx}{dt} = \sqrt{3x+1}$

$\therefore \frac{dt}{dx} = \frac{1}{\sqrt{3x+1}}$

$\therefore t = \int (3x+1)^{-\frac{1}{2}}\,dx \longleftarrow$ 1 mark

$\therefore t = \frac{2(3x+1)^{\frac{1}{2}}}{3} + c$

when $t=0,\ x=0$ $\quad \therefore c = -2/3$

$\therefore t = \frac{2(3x+1)^{\frac{1}{2}}}{3} - \frac{2}{3} \longleftarrow$ 1 mark

when $t=5$ $\quad 5 = \frac{2\sqrt{3x+1}}{3} - \frac{2}{3}$

$\therefore 15 = 2\sqrt{3x+1} - 2$

$\therefore \sqrt{3x+1} = 17/2$

$\therefore x = 95/4 \longleftarrow$ 1 mark

Question Five

a) $2\cos\left(2t + \frac{\pi}{6}\right)$

$= 2\cos 2t \cos\frac{\pi}{6} - 2\sin 2t \sin\frac{\pi}{6}$

$= 2(\cos 2t)\left(\frac{\sqrt{3}}{2}\right) - 2(\sin 2t)\frac{1}{2}$

$= \sqrt{3}\cos 2t - \sin 2t$

1 mark

b) i) $v = -2\sqrt{3}\sin 2t - 2\cos 2t$

$\therefore \ddot{x} = -4\sqrt{3}\cos 2t + 4\sin 2t \leftarrow$ 1 mark

$= -4(\sqrt{3}\cos 2t - \sin 2t)$

But $\sqrt{3}\cos 2t - \sin 2t = x - 5$

$\therefore \ddot{x} = -4(x-5) \longleftarrow$ 1 mark

ii) C.O.M $= 5$ and Amp $= \cancel{4}2$

$\therefore$ Particle is oscillating between

$x = 1$ and $x = 9$

3 7

1 mark

Question Six

a) $T = B + Ae^{-kt}$

when $t = 0$, $T = 90$ and $B = 30$

$\therefore A = 60$ ⟵ 1 mark

when $t = 2$, $T = 80$

$\therefore 80 = 30 + 60e^{-2k}$

$\therefore e^{-2k} = \frac{5}{6}$

$\therefore -2k = \ln(5/6)$

$\therefore k = \frac{\ln(5/6)}{-2}$ or $\frac{1}{2}\ln\left(\frac{6}{5}\right)$ ⟵ 1 mark

) when $t = 7$

$T = 30 + 60e^{-7k}$

$\therefore T = 62°$ (nearest degree) ⟵ 1 mark

Question Seven

a) $\ddot{x} = -4(x+1)$

1 mark 1 mark

$P = \pi$ secs

$\therefore \frac{2\pi}{n} = \pi$

$\therefore n = 2$

b) Max. acc. at the extremities ie. when $x = -5$ (or 3)

$\therefore \ddot{x} = -4(-4)$

$\therefore \ddot{x} = 16$

$\therefore$ Max. acc. is $16\ ms^{-2}$ ⟵ 1 mark

c) $\ddot{x} = -4x - 4$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x - 4$

$\therefore \frac{1}{2}v^2 = -2x^2 - 4x + C$ ⟵ 1 mark

when $x = 3$, $v = 0$ $\therefore C = 30$

$\therefore v^2 = -4x^2 - 8x + 60$ ⟵ 1 mark

Max. speed at C.O.M. ie. when $x = -1$

$\therefore v^2 = -4 + 8 + 60$

$\therefore v^2 = 64$

$\therefore v = \pm 8\ ms^{-1}$

$\therefore$ Max. speed is $8\ ms^{-1}$ ⟵ 1 mark

Question Eight

a) Area of sector $= \frac{1}{2}r^2\theta$ $\quad\therefore \frac{1}{2}$ Area of sector $= \frac{r^2\theta}{4}$

and Area of triangle $= \frac{1}{2}r^2\sin\theta$ $\quad = \frac{1}{2}r^2\frac{\theta}{2}$

$\therefore$ Area of $\Delta = \frac{1}{2}$ Area of sector if $\sin\theta = \frac{\theta}{2}$

ie. AB bisects the sector if $\sin\theta = \frac{\theta}{2}$

1 mark

b) let $P(\theta) = \sin\theta - \frac{\theta}{2}$

Now $P(1\cdot8) = 0\cdot07384763$

and $P(2) = -0\cdot090702573$

$\therefore$ $\theta = 1\cdot8$ is the better approx.

1 mark

c) $a_1 = a_0 - \dfrac{P(a_0)}{P'(a_0)}$

1 mark

$P'(\theta) = \cos\theta - \frac{1}{2}$

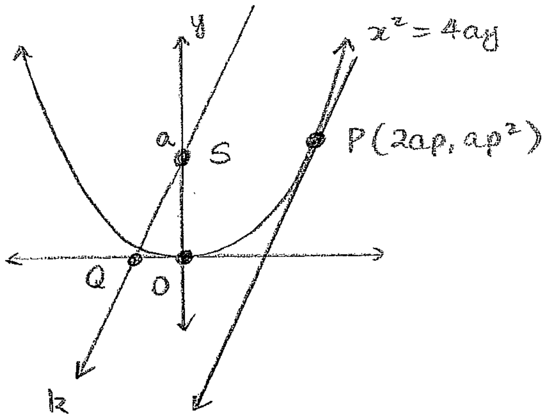
$\therefore P'(1\cdot8) = \cos 1\cdot8 - \frac{1}{2}$

$= -0\cdot727202094$

$\therefore a_1 = 1\cdot8 + \dfrac{0\cdot07384763}{0\cdot727202094}$

$\therefore a_1 = 1\cdot90$ (2dp) $\longleftarrow$ 1 mark

Question Nine



a) $y = \frac{x^2}{4a}$

$\therefore y' = x/2a$

$\therefore y'(2ap) = p$

$\therefore$ m of T at P $= p$

$\therefore$ m of k $= p$

Eqn of T at P: $y - a = p(x - 0)$

$\therefore px - y + a = 0$ $\longleftarrow$ 1 mark

b) let $y = 0$ $\quad\therefore x = -a/p$

$\therefore$ Coords of Q are $(-a/p, 0)$ $\longleftarrow$ 1 mark

Mpt of QS $= \left(\dfrac{-a/p}{2}, \dfrac{a}{2}\right)$ $\longleftarrow$ 1 mark

c) $x = -\dfrac{a}{2p}$ and $y = \dfrac{a}{2}$

$\therefore$ LOCUS OF M is $y = a/2$ $\longleftarrow$ 1 mark

Question 10

a) i) $$\text{LHS} = -2\cos\left(x+\tfrac{\pi}{6}\right)$$
$$= -2\left[\cos x \cos\tfrac{\pi}{6} - \sin x \sin\tfrac{\pi}{6}\right]$$
$$= -2\left[\frac{\sqrt{3}\cos x}{2} - \frac{\sin x}{2}\right]$$
$$= -\sqrt{3}\cos x + \sin x$$
$$= \sin x - \sqrt{3}\cos x$$
$$= \text{RHS}$$

1 mark

ii) $$\text{LHS} = \tan^2 x - 3$$
$$= \frac{\sin^2 x}{\cos^2 x} - 3$$
$$= \frac{\sin^2 x - 3\cos^2 x}{\cos^2 x}$$
$$= \text{RHS}$$

1 mark

b) $$\lim_{x\to\frac{\pi}{3}} \frac{\tan^3 x - 3\tan x}{\cos\left(x+\frac{\pi}{6}\right)} = \lim_{x\to\frac{\pi}{3}} \frac{\tan x(\tan^2 x - 3)}{-\frac{1}{2}\left[-2\cos\left(x+\frac{\pi}{6}\right)\right]}$$

1 mark for correct subst to this line

$$= \lim_{x\to\pi/3} \frac{-2\tan x(\tan^2 x - 3)}{\sin x - \sqrt{3}\cos x}$$

$$= \lim_{x\to\frac{\pi}{3}} \frac{\frac{-2\sin x}{\cos x}\cdot\frac{\sin^2 x - 3\cos^2 x}{\cos^2 x}}{\sin x - \sqrt{3}\cos x}$$

continued next page ...

Q10 continued.

$$= \lim_{x\to\frac{\pi}{3}} \frac{\frac{-2\sin x}{\cos^3 x}(\sin x - \sqrt{3}\cos x)(\sin x + \sqrt{3}\cos x)}{\sin x - \sqrt{3}\cos x}$$

1 mark for diff. of 2 squares

$$= \lim_{x\to\frac{\pi}{3}} \frac{-2\sin x}{\cos^3 x}(\sin x + \sqrt{3}\cos x)$$

$$= \frac{-2\sin\frac{\pi}{3}}{\cos^3\frac{\pi}{3}}\left(\sin\tfrac{\pi}{3} + \sqrt{3}\cos\tfrac{\pi}{3}\right)$$

$$= \frac{-2\left(\frac{\sqrt{3}}{2}\right)}{\frac{1}{8}}\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)$$

$$= -24$$

1 mark for correct answer

Question Eleven

a) $\ddot{x} = 0$

$\therefore \dot{x} = C$

when $t=0$, $\dot{x} = V\cos\theta$

$\therefore C = V\cos\theta$

$\therefore \dot{x} = V\cos\theta$

Now $x = Vt\cos\theta + D$

when $t=0$, $x=0$ $\therefore D=0$

$\therefore x = Vt\cos\theta$

since $V=20$, $x = 20t\cos\theta$

$\ddot{y} = -g$

$\therefore \dot{y} = -gt + k$

when $t=0$, $\dot{y} = V\sin\theta$

$\therefore k = V\sin\theta$

$\therefore \dot{y} = -gt + V\sin\theta$

Now $y = -\frac{gt^2}{2} + Vt\sin\theta + M$

when $t=0$, $y=0$ $\therefore M=0$

$\therefore y = -\frac{gt^2}{2} + Vt\sin\theta$

since $g=10$ and $V=20$

$y = -5t^2 + 20t\sin\theta$

1 mark each for correctly deriving eqns of motion

b) when $x=20$

$t = \frac{1}{\cos\theta}$

subst $t = \frac{1}{\cos\theta}$ into $y = -5t^2 + 20t\sin\theta$

$\therefore y = -\frac{5}{\cos^2\theta} + \frac{20\sin\theta}{\cos\theta}$ 1 mark

$\therefore y = 20\tan\theta - 5\sec^2\theta$

But when $t = \frac{1}{\cos\theta}$, $y = h$

$\therefore h = 20\tan\theta - 5\sec^2\theta$

c) P.T.O.

Q11 continued

c) $h = 20\tan\theta - 5(\cos\theta)^{-2}$

$\therefore \frac{dh}{d\theta} = 20\sec^2\theta + 10(\cos\theta)^{-3}.(-\sin\theta)$

$= 20\sec^2\theta - \frac{10\sin\theta}{\cos^3\theta}$

$= 20\sec^2\theta - 10\tan\theta\sec^2\theta$ 1 mark

$= 10\sec^2\theta(2-\tan\theta)$

d) let $\frac{dh}{d\theta} = 0$

$\therefore 10\sec^2\theta(2-\tan\theta) = 0$

$\therefore 2 - \tan\theta = 0$

$\therefore \tan\theta = 2$ 1 mark

$\therefore \theta = \tan^{-1}(2)$

θ	1	2	3
$\frac{dh}{d\theta}$	/	—	\

$\therefore$ MAX! ← 1 mark

Now $h = 20\tan\theta - 5\sec^2\theta$

$\therefore h = 20(2) - 5(1+4)$ ← (since $\sec^2\theta = 1 + \tan^2\theta$)

$= 40 - 25$

$\therefore h = 15$ m ← 1 mark

$\therefore$ Max. height the ball can reach up the wall is 15 m