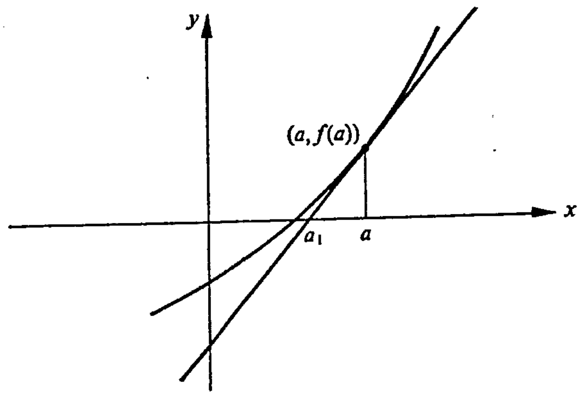
NEWTON'S METHOD OF APPROX

Exercises

1. (a) Show that the equation $x^2 - 3x - 1 = 0$ has a root between $x = 3$ and $x = 4$.
   (b) Use the method of halving the interval to show that the root is between $x = 3.25$ and $x = 3.5$.
2. Show that the polynomial equation
$$2x^3 - 3x^2 - 12x + 15 = 0$$
has a root between $x = 1$ and $x = 2$. Use the method of halving the interval to show that the root lies between $x = 1$ and $x = 1.25$.
3. Sketch $y = 2x^5 - 1$, showing any turning points. From the graph, the root of $2x^5 - 1 = 0$ lies between which 2 integers? Halve the interval twice to find an approximation to the root correct to 2 decimal places.
4. The polynomial equation $3x^7 - 1 = 0$ has a root between $x = 0$ and $x = 1$. Halve the interval twice to find an approximation to the root.

Newton's Method



If $x = a$ is close to the root of the equation $f(x) = 0$, the x-intercept of the tangent at a (i.e. a_1) is closer to the root.

$$a_1 = a - \frac{f(a)}{f'(a)}$$

Notes (a) This method gives a more accurate solution in fewer steps than halving the interval.
(b) Newton's Method doesn't work in some cases, (e.g. if a is too far away from the root or $f'(a) = 0$).

EXAMPLE

Find an approximation of the root of
$$y = x^3 + 2x - 1$$
by using Newton's Method once and starting with an approximation of $x = 0.5$.

Solution
$$f(0.5) = 0.5^3 + 2(0.5) - 1 = 0.125.$$
$$f'(x) = 3x^2 + 2$$
$$f'(0.5) = 3(0.5)^2 + 2 = 2.75.$$
$$a_1 = a - \frac{f(a)}{f'(a)} = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.5 - \frac{0.125}{2.75} \doteqdot 0.455.$$

Exercises

5. The polynomial equation $x^3 - 3x^2 + 9 = 0$ has a root near $x = -1.5$. Find an approximation of the root (to 2 decimal places) by using 1 application of Newton's Method.
6. A root of the equation $x^3 - 6x^2 + 12x - 1 = 0$ lies near $x = 0$. Use 1 application of Newton's Method to find an approximation of the root, to 2 significant figures.
7. (a) Show that a root of the equation $x^4 - 3x^2 + 5x + 1 = 0$ lies between $x = -1$ and $x = 0$.
   (b) Use 1 application of Newton's Method to find an approximation to the root, using $x = -0.5$ as the first approximation.
8. The equation $e^{2x} - 2 = 0$ has a root between $x = 0$ and $x = 1$. Find an approximation to the root by using 2 applications of Newton's Method.

Test yourself

Revision questions

9. Show that $x^3 - 3x^2 + 3 = 0$ has a root between $x = 1$ and $x = 2$. By starting with an approximation of $x = 1.5$, use 2 applications of Newton's Method to find an approximation of the root to 3 significant figures.

10. The equation $2x^5 - 3x^2 + 6 = 0$ has a root between $x = -2$ and $x = -1$.
    (a) Use Newton's Method to find an approximation to the root.
    (b) Use halving the interval twice to find an approximation to the root.
    (c) Which is the best approximation?

11. (a) Sketch the curve $y = -x^3 - 3x^2 + 9x + 1$ showing any stationary points.
    (b) The equation $-x^3 - 3x^2 + 9x + 1 = 0$ has 3 roots. Use Newton's Method to find an approximation to each root.
    (c) One of these is not a good approximation. Which one?

12. The equation $\sin x = 2 - 2x$ has a root between $x = 0$ and $x = 1$. Use $f(x) = \sin x - 2 + 2x$ and Newton's Method to find an approximation of the root, correct to 2 decimal places.

13. (a) Sketch the curve $y = 2x^3 - 3x^2 + 2$, showing any stationary points.
    (b) From your graph, how many roots of $2x^3 - 3x^2 + 2 = 0$ are there?
    (c) By halving the interval twice, find an approximation to the root.

Challenge questions

14. (a) Find the equation of the tangent to the curve $y = f(x)$ at the point where $x = a$.
    (b) Find the x-intercept of the tangent and hence prove that
    $$a_1 = a - \frac{f(a)}{f'(a)}$$
    where a_1 is the x-intercept of the tangent.

15. Sketch the curve $y = x^4 - 6x^2 + 8x + 1$ showing any turning points. Estimate the roots of equation $x^4 - 6x^2 + 8x + 1 = 0$ by choosing appropriate approximations for the roots and using 1 application of Newton's Method.

16. Why is the value $x = 1$ a poor estimate of the root of $x^2 - 2x - 1 = 0$ when using Newton's Method? Illustrate with a sketch.

17. (a) Divide $p(x) = x^4 - 4x^3 + 4x^2 + x - 2$ into its prime factors.
    (b) Sketch the curve on the number plane. How many roots does the equation $x^4 - 4x^3 + 4x^2 + x - 2 = 0$ have?
    (c) Using Newton's Method to find approximations to the roots where necessary, find all the roots.

18. (a) Show that $f(x) = x^4 - 10$ has a root between $x = 1$ and $x = 2$.
    (b) Use the method of halving the interval to show that $\sqrt[4]{10}$ lies between 1.75 and 1.875.

Polynomials 2

Estimation of roots (3 unit)

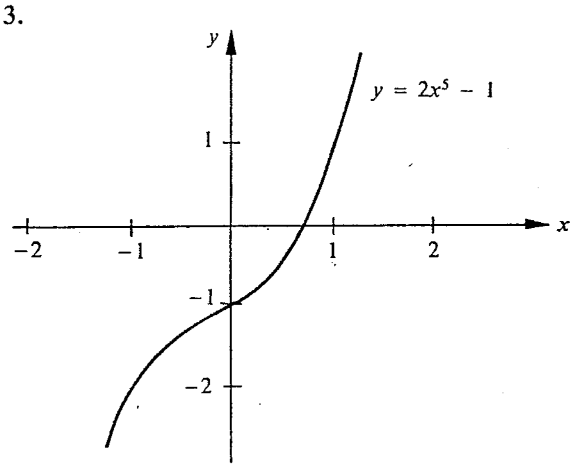
1. (a) Show $f(3) < 0$, $f(4) > 0$.
(b) Show $f(3.25) < 0$, $f(3.5) > 0$.

2. Let $f(x) = 2x^3 - 3x^2 - 12x + 15$
$f(1) = 2 > 0$ and $f(2) = -5 < 0$,
so there is a root between $x = 1$ and 2.

$f\left(\frac{1+2}{2}\right)$
$= f(1.5)$
$= 2(1.5)^3 - 3(1.5)^2 - 12(1.5) + 15$
$= -3$,
$\therefore$ the root lies between $x = 1$ and $x = 1.5$.

$f\left(\frac{1+1.5}{2}\right)$
$= f(1.25)$
$= 2(1.25)^3 - 3(1.25)^2 - 12(1.25) + 15$
$\doteqdot -0.78$,
$\therefore$ the root lies betweeen $x = 1$ and $x = 1.25$.

3.



The root lies between $x = 0$ and $x = 1$.
The root is approximately 0.88.

4. $f(0) = 3(0)^7 - 1 = -1$
$f(1) = 3(1)^7 - 1 = 2$
$\therefore f(0) < 0,\ f(1) > 0$
$f(0.5) = 3(0.5)^7 - 1 \doteqdot -0.977 < 0$
$\therefore$ root lies between $f(0.5)$ and $f(1)$.
$f(0.75) = 3(0.75)^7 - 1 \doteqdot -0.5996$
$\therefore$ root lies between $f(0.75)$ and $f(1)$.
$f(0.75)$ gives a closer approximation to 0,
$\therefore x \doteqdot 0.75$

5. -1.43

6. Let $f(x) = x^3 - 6x^2 + 12x - 1$.
Then $f'(x) = 3x^2 - 12x + 12$

$$a_1 = a - \frac{f(a)}{f'(a)} = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{-1}{12} \doteqdot 0.083$$

7. (a) Show $f(-1) < 0$, $f(0) > 1$.
(b) $x \doteqdot -0.21$

8. $f(x) = e^{2x} - 2$
$f'(x) = 2e^{2x}$
Using $x = 0.5$,
$f(0.5) = e^{2\times0.5} - 2 \doteqdot 0.72$
$f'(0.5) = 2e^{2\times0.5} \doteqdot 5.44$

$$a_1 = a - \frac{f(a)}{f'(a)} = 0.5 - \frac{0.72}{5.44} \doteqdot 0.37$$

$f(0.37) = e^{2\times0.37} - 2 \doteqdot 0.087$
$f'(0.37) = 2e^{2\times0.37} \doteqdot 4.17$

$$a_1 = a - \frac{f(a)}{f'(a)} = 0.37 - \frac{0.087}{4.17} \doteqdot 0.35$$

Revision questions

9. $f(1) = 1 > 0$
$f(2) = -1 < 0$
$\therefore$ the root lies between $x = 1$ and $x = 2$.
$\alpha \doteqdot 1.35$

10. (a) Using $x = -1.5$,
$f(-1.5) = 2(-1.5)^5 - 3(-1.5)^2 + 6 = -15.9375$
$f'(x) = 10x^4 - 6x$
$f'(-1.5) = 10(-1.5)^4 - 6(-1.5) = 59.625$

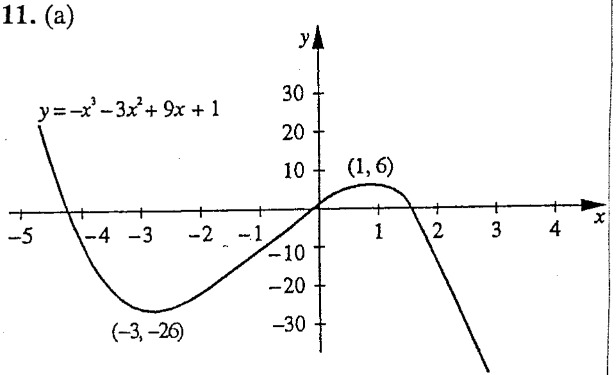
$$a_1 = a - \frac{f(a)}{f'(a)} = -1.5 - \frac{f(-1.5)}{f'(-1.5)} = -1.5 - \frac{-15.9375}{59.625} = -1.23$$

(b) $f(-1) = 2(-1)^5 - 3(-1)^2 + 6 = 1$
$f(-2) = 2(-2)^5 - 3(-2)^2 + 6 = -70$

Since $f(-2) < 0$, $f(-1) > 0$, the root lies between them.
$f(-1.5) = -15.9375$ from (a)
Since $f(-1.5) < 0$, $f(-1) > 0$, the root lies between them.
$f(-1.25) = 2(-1.25)^5 - 3(-1.25)^2 + 6$
$= -4.79$
Since $f(-1.25) < 0$, $f(-1) > 0$, the root lies between them.
Since $f(-1)$ is closer to zero, $x = -1$ is the closest approximation.

(c) $f(-1.23) = 2(-1.23)^5 - 3(-1.23)^2 + 6$
$= -4.25$
$f(-1) = 1$
$\therefore x = -1$ gives the best approximation since $f(-1)$ is closer to 0.

11. (a)



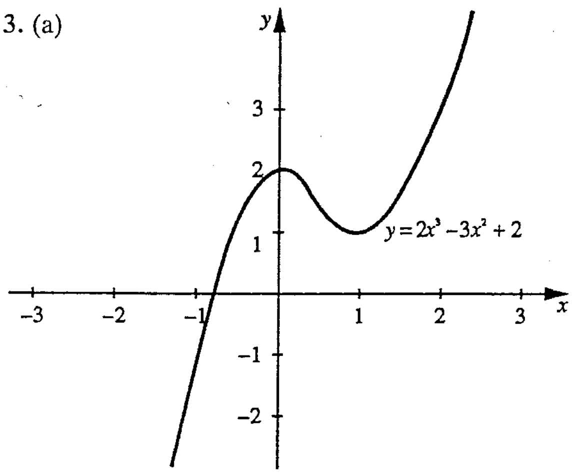
(b) Using $a = -4.5$, $x \doteqdot -4.87$
Using $a = -0.5$, $x \doteqdot -0.13$
Using $a = 1.5$, $x \doteqdot 2.15$

(c) $x \doteqdot 2.15$ is not a good approximation.

12. $f(0.5) = \sin 0.5 - 2 + 2 \times 0.5$
$= -0.52$
$f'(x) = \cos x + 2$
$f'(0.5) = \cos 0.5 + 2$
$\doteqdot 2.88$

$$a_1 = a - \frac{f(a)}{f'(a)}$$
$$= 0.5 - \frac{-0.52}{2.88}$$
$$= 0.68$$

13. (a)



(b) There is 1 root, between $x = -1$ and $x = 0$.
(c) $x \doteqdot -0.75$

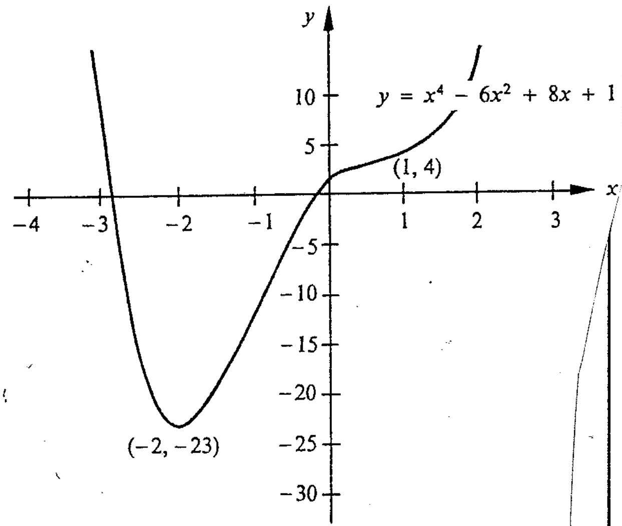
Challenge questions

14. (a) When $x = a, y = f(a)$
When $x = a, y' = f'(a)$
Equation:
$y - y_1 = m(x - x_1)$
$y - f(a) = f'(a)(x - a)$

(b) For x-intercept, $y = 0$.
$\therefore 0 - f(a) = f'(a)(x - a)$
$-f(a) = f'(a)x - f'(a)a$
$f'(a)a - f(a) = f'(a)x$

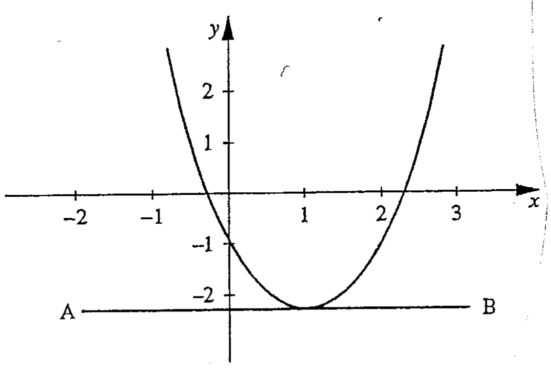
$$\frac{f'(a)a - f(a)}{f'(a)} = x$$
$$a - \frac{f(a)}{f'(a)} = x$$
$$\therefore a_1 = a - \frac{f(a)}{f'(a)}$$ where a_1 is the x-intercept.

15.



$\alpha \doteqdot -2.94$, $\beta \doteqdot -0.13$

16. Sketching $y = x^2 - 2x - 1$:
$y' = 2x - 2$
For stationary points, $y' = 0$
i.e. $2x - 2 = 0$
$2x = 2$
$x = 1$
When $x = 1, y = 1^2 - 2 \times 1 - 1$
$= -2$
$y'' = 2 > 0 \therefore$ minimum



The tangent at $x = 1$ is horizontal, so AB has no x-intercept.

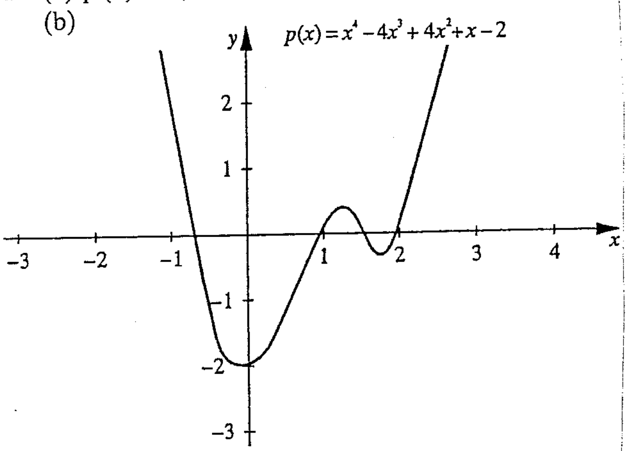
$f'(1) = 0$, so $a_1 = 1 - \dfrac{f(1)}{f'(1)}$ is undefined.

Also, $x = 1$ is not near any root (one lies between 2 and 3, and the other between -1 and 0).

For all of these reasons, $x = 1$ is a poor estimate.

17. (a) $p(x) = (x - 2)(x - 1)(x^2 - x - 1)$

(b)



(c) $x = 1, 2$

Using $f(1.5)$, $x \doteqdot 1.625$.

Using $f(-0.5)$, $x \doteqdot -0.64$.

18. (a)
$$\begin{aligned} f(1) &= 1^4 - 10 \\ &= -9 \\ &< 0 \\ f(2) &= 2^4 - 10 \\ &= 6 \\ &> 0 \end{aligned}$$

$\therefore$ root lies between $x = 1$ and $x = 2$.

(b) If
$$\begin{aligned} x^4 - 10 &= 0 \\ x^4 &= 10 \\ x &= \sqrt[4]{10} \end{aligned}$$

$\therefore$ root of $f(x) = x^4 - 10$ will give an approximate value of $\sqrt[4]{10}$.

$$\begin{aligned} f(1.5) &= 1.5^4 - 10 \\ &= -4.9375 \\ &< 0 \end{aligned}$$

$\therefore$ root lies between $x = 1.5$ and $x = 2$.

$$\begin{aligned} f(1.75) &= 1.75^4 - 10 \\ &= -0.621 \\ &< 0 \end{aligned}$$

$\therefore$ root lies between $x = 1.75$ and $x = 2$.

$$\begin{aligned} f\left(\frac{1.75 + 2}{2}\right) &= f(1.875) \\ &= 1.875^4 - 10 \\ &= 2.36 \\ &> 0 \end{aligned}$$

$\therefore$ root lies between $x = 1.75$ and $x = 1.875$.