NORTH SYDNEY GIRLS HIGH SCHOOL

YEAR 12 – TERM 1 ASSESSMENT

2006

MATHEMATICS EXTENSION 1

TIME ALLOWED: 60 minutes
Plus 2 minutes reading time

INSTRUCTIONS:

- Start each question on a new page
- Hand each question in separately, including a sheet for non-attempts
- Show all necessary working

This task is worth 20% of the HSC Assessment Mark

Question One (9 Marks) — Marks

(a) What is the exact value of $\cos\left(\frac{\pi}{6}\right)$? — 1

(b) Differentiate $\cos(x^2+1)$ — 1

(c) Find $\int \sec^2 5x \, dx$ — 1

(d) (i) Sketch the curve $y = 4\sin 2x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ — 2

(ii) On your diagram for part (i), sketch the line $y = \frac{1}{3}x$, and shade the region represented by $\int_0^{\frac{\pi}{4}} \left(4\sin 2x - \frac{1}{3}x\right) dx$ — 2

(iii) Find the exact value of the integral in part (ii). — 2

Question Two (8 Marks)

(a) A sphere is being heated so that its surface area is increasing at a constant rate of 25 cm^2 per second. Find the rate of increase of the volume when the radius is 5 cm. — 3

(b) One hundred grams of cane sugar in water are being converted into dextrose at a rate which is proportional to the amount unconverted at any time,
i.e. if M grams are converted in t minutes, then $\frac{dM}{dt} = k\,(100 - M)$,
where k is a constant.

(i) Show that $M = 100 + Ae^{-kt}$, where A is a constant which satisfies the above equation. — 2

(ii) Find the value of A (initially no cane sugar has been converted to dextrose) — 1

(iii) If 40 grams are converted in the first 15 minutes, find how many grams are converted in the first 30 minutes. — 2

Question Three (10 Marks)

(a) Evaluate $\lim_{x \to 0} \frac{\sin\left(\frac{x}{4}\right)}{3x}$ — 2

(b) (i) Express $\cos x - \sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$ — 2

(ii) Hence, or otherwise, solve the equation $\cos x - \sin x = \frac{\sqrt{2}}{2}$ for $0 \le x \le 2\pi$ — 2

(c) Prove $\frac{\tan A}{\tan 2A - \tan A} \equiv \cos 2A$ — 4

Question Four (10 Marks)

(a) Find the exact value of $\sin 105°$ **2**

(b) Find the volume of the solid generated when $y = \sin 3x$ is rotated around the x axis from $x = 0$ to $x = \frac{\pi}{3}$. **4**

(c) Differentiate $x \sin 3x$ with respect to x and hence evaluate $\int_0^{\frac{\pi}{2}} x\cos 3x\,dx$ **4**

Question Five (9 Marks)

(a) If $y = \tan 2x$, find the equation of the tangent to the curve at $x = \frac{\pi}{6}$ **3**

(b) Find the acute angle between the lines $4x + y + 5 = 0$ and $6x + 3y - 7 = 0$ correct to the nearest minute. **3**

(c) Solve the equation $\sin 2\theta + \cos\theta = 0$ for $0 \le \theta \le 2\pi$ **3**

Question Six (10 Marks)

(a) Solve $5\sin\theta - 2\cos\theta = 2$ for $0° \le \theta \le 360°$, using the result that $\tan\frac{\theta}{2} = t$ **4**

(b) A particle moves along a straight line so that its displacement, x metres, from a fixed point O is given by $x = 1 - 3\cos\left(\frac{t}{2}\right)$, where t is measured in seconds.

(i) Sketch the graph of x as a function of t for $0 \le t \le 4\pi$ **2**

(ii) Hence, or otherwise, find when and where the particle first comes to rest after $t = 0$ **2**

(iii) Find a time when the particle reaches its maximum speed. What is this speed? **2**

END OF TEST

Question 1

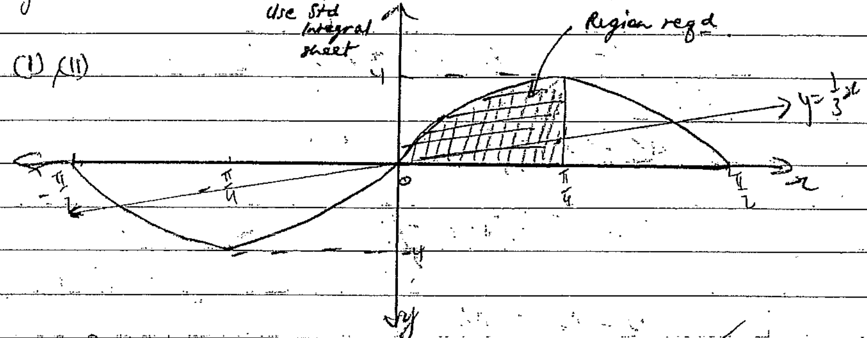
a) $\frac{\sqrt{3}}{2}$ ✓

b) $\frac{d(\cos(x^2+1))}{dx} = -2x\sin(x^2+1)$ ✓

c) $\int \sec^2 5x \, dx = \frac{1}{5}\tan 5x + c$ ✗

Use std integral sheet

d) (i) (ii)



(iii) $\int_0^{\frac{\pi}{4}} (4\sin 2x - \frac{1}{3}x)\, dx = \left[-2\cos 2x - \frac{1}{6}x^2\right]_0^{\frac{\pi}{4}}$

$= -\frac{\pi^2}{96} + 2$

$= 2 - \frac{\pi^2}{96}$ ✓

Question 2

a) $\frac{dA}{dt} = 25\text{ cm}^2$

$V = \frac{4}{3}\pi r^3$ $\quad A = 4\pi r^2$

$\frac{dV}{dr} = 4\pi r^2$ $\quad \frac{dA}{dr} = 8\pi r$

$\frac{dU}{dr} = \frac{dU}{dt} \times \frac{dt}{dr}$

$\frac{dV}{dA} = \frac{dV}{dr} \times \frac{dr}{dA}$

$= 4\pi r^2 \times \frac{1}{8\pi r}$

$= \frac{r}{2}$

$\therefore \frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$

$= \frac{r}{2} \times 25 = \frac{5}{2} \times 25 \text{ cm}^3/s = \frac{125}{2}\text{ cm}^3/s$

$= 62\frac{1}{2}\text{ cm}^3/s$.

Question 2

b) (i) $\frac{dM}{dt} = 100k - Mk$

$\frac{dt}{dM} = \frac{1}{100k - Mk}$

$t = -\frac{1}{k}\ln(100 - M) + c$

$t - c = -\frac{1}{k}\ln(100 - M)$

$k(c - t) = \ln(100 - M)$

$e^{kc - kt} = 100 - M$ $\quad$ let $A = e^{kc}$

$100 - Ae^{-kt} = M$

(ii) $0 = 100 - A$

$A = 100$

(iii) $40 = 100 - 100e^{-15k}$

$60 = 100e^{-15k}$

$\ln\frac{6}{10} = -15k$

$k = \frac{\ln\frac{6}{10}}{-15}$

for $t = 30$

$M = 100 - 100e^{-k30}$

$= 64$

64 gms are converted.

Question 3

a) $\lim_{x\to 0} \frac{\sin(\frac{x}{4})}{3x} = \frac{1}{3}\lim_{x\to 0} \frac{1}{4}\frac{\sin(\frac{x}{4})}{\frac{x}{4}}$

$= \frac{1}{12}\lim_{x\to 0} \frac{\sin(\frac{x}{4})}{\frac{x}{4}}$ ✓

$= \frac{1}{12}$ ✓

b) (i) $\cos x - \sin x \Rightarrow \cos\alpha = 1 \quad \sin\alpha = 1 \quad \therefore R = \sqrt{2}$ ✓

$\alpha = \frac{\pi}{4}$ ✓

$\therefore \cos x - \sin x = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$ ✓

(ii) $\cos x - \sin x = \frac{\sqrt{2}}{2}$

$\sqrt{2}\cos\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\cos\left(x + \frac{\pi}{4}\right) = \frac{1}{2}$ ✓

$-\frac{\pi}{4} \le x \le \frac{7\pi}{4}$

$x + \frac{\pi}{4} = -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3} \quad \therefore 0 \le x + \frac{\pi}{4} \le 2\pi$

$x = -\frac{\pi}{12}, \frac{17\pi}{12}$ ✓

c) $LHS = \frac{\tan A}{\tan 2A - \tan A} = \frac{\tan A}{\frac{2\tan A}{1-\tan^2 A} - \tan A}$

$= \frac{\tan A}{\frac{2\tan A - \tan A(1-\tan^2 A)}{1-\tan^2 A}}$ ✓

$= \frac{1-\tan^2 A}{2 - 1 + \tan^2 A}$ ✓

$= \frac{1-\tan^2 A}{1-\tan^2 A}$

Question 4

a) $\sin(105°) = \sin(60+45)$

$= \sin 60\cos 45 + \sin 45\cos 60$

$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$ ✓

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

b) $V = \pi\int_0^{\frac{\pi}{3}} \sin^2 3x \, dx$

$= \pi\int_0^{\frac{\pi}{3}} \frac{1-\cos 6x}{2} \, dx$ ✓

$= \frac{\pi}{2}\left[x - \frac{\sin 6x}{6}\right]_0^{\frac{\pi}{3}}$

$= \frac{\pi^2}{6}$ ✓

c) $\frac{d(x\sin 3x)}{dx} = \sin 3x + 3x\cos 3x$ ✓ $\quad u = x \quad v = \sin 3x$ $\quad u' = 1 \quad v' = 3\cos 3x$

$\therefore \int_0^{\frac{\pi}{2}} x\cos 3x = \frac{1}{3}\int_0^{\frac{\pi}{2}} 3x\cos 3x + \sin 3x - \sin 3x$

$= \frac{1}{3}\left[x\sin 3x + \frac{1}{3}\cos 3x\right]_0^{\frac{\pi}{2}}$ ✓

$= \frac{1}{3}\left[-\frac{\pi}{2} - \frac{1}{3}\right]$

$= -\left(\frac{\pi}{6} + \frac{1}{9}\right)$ ✓

Question 5

(a) $y = \tan 2x$

$y' = 2\sec^2 2x$

at $(\frac{\pi}{6}, \sqrt{3})$ $\therefore y' = 8$ ✓

$\therefore y - \sqrt{3} = 8(x - \frac{\pi}{6})$

$8x - y - \frac{4\pi}{3} + \sqrt{3} = 0$ ✓

(b) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ $m_1 = -4$, $m_2 = -2$

$= \left|\frac{-2}{9}\right|$

$\theta \approx 12^\circ 32'$

(c) $\sin 2\theta + \cos\theta = 0$

$2\sin\theta\cos\theta + \cos\theta = 0$

$\cos\theta(2\sin\theta + 1) = 0$

$\therefore \cos\theta = 0$ $\quad \sin\theta = -\frac{1}{2}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ ✓ $\quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$ ✓

$\therefore \theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$ ✓

Question 6

(a) $5\sin\theta - 2\cos\theta = 2$ $\quad$ let $\tan\frac{\theta}{2} = t$ $\quad \therefore \sin\theta = \frac{2t}{1+t^2}$

$\frac{10t}{1+t^2} - \frac{(2-2t^2)}{1+t^2} = 2$ $\quad \cos\theta = \frac{1-t^2}{1+t^2}$

$\frac{10t - 2 + 2t^2}{1+t^2} = 2$

Quest 6

(a) cont'd. $\frac{2t^2 + 10t - 2}{1+t^2} = 2$

$2t^2 + 10t - 2 = 2 + 2t^2$ $\quad 10t = 2$

$2t^2 - 10t + 4 = 0$ $\quad t = \frac{1}{5}$

$t = \frac{10 \pm \sqrt{100 - 32}}{4}$ $\quad \therefore \tan\frac{\theta}{2} = \frac{1}{5}$ $\quad 0^\circ \le \frac{\theta}{2} \le 18$

$\frac{\theta}{2} = 11^\circ 19'$

$= \frac{10 \pm \sqrt{68}}{4}$ $\quad \theta = 22^\circ 38'$

Also need to test for $\theta = 180^\circ$

$= \frac{10 \pm 2\sqrt{17}}{4}$ $\quad 5\sin\theta - 2\cos\theta = 2$

$= \frac{5 \pm \sqrt{17}}{2}$ $\quad 0 - 2(-1) = 2$

is true

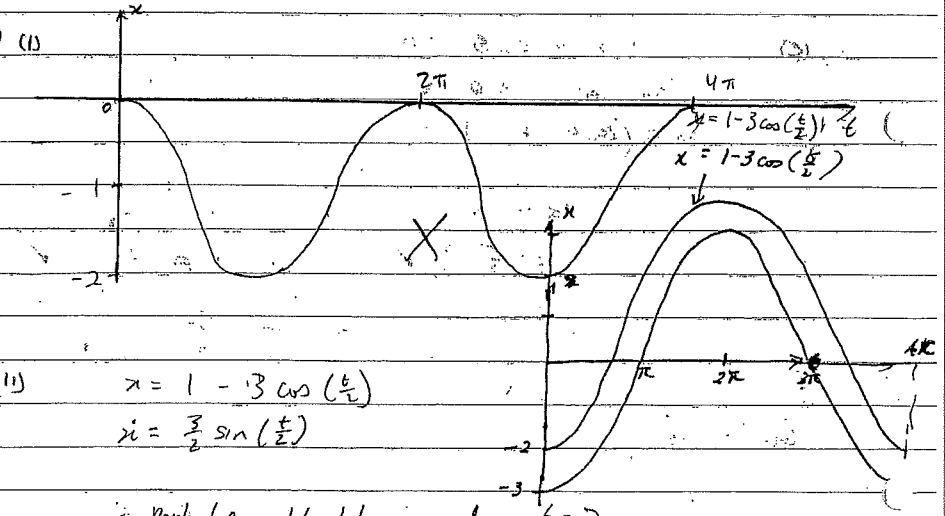
$\therefore \theta = 180^\circ$ is also a soln.

$\therefore \theta \approx 155^\circ 16'$

$47^\circ 21'$

180°

(b) (i)



(ii) $x = 1 - 3\cos(\frac{t}{2})$

$\dot{x} = \frac{3}{2}\sin(\frac{t}{2})$

$\therefore$ particle 1st stationary at $t = 2\pi$

$x = 4$

For $\dot{x} = 0$, $t = 2\pi$

Qu 6

b) (iii) max speed at $\ddot{x} = 0$ i.e. at point of inflexion

at $t = \pi$ or 3π.

$\ddot{x} = \frac{3}{4}\cos\left(\frac{t}{2}\right) = 0$

$\therefore t = \pi$ ✓ the speed is $\frac{3}{2}$ m/s. ✓