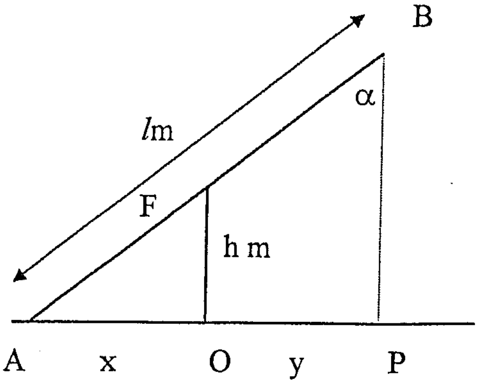
Rates of Change Name: ______________________ Total: /10

OF is a vertical fence, height h m, standing on horizontal ground.

A pole, AB, of length l m, rests across the fence.
P is the point of the ground vertically below B.
OA = x m, OP = y m and <ABP = α.



1. Prove that $x = h \tan \alpha$

2. Prove the $y = l \sin \alpha - h \tan \alpha$.

3. By considering $\frac{dy}{d\alpha}$ and $\frac{dx}{d\alpha}$, prove that $\frac{dy}{dx} = \frac{l}{h} \cos^3 \alpha - 1$.

The end A of the pole is moved along the ground, away from the fence, at a constant rate of 8m / min. If $l = 6.75$ and $h = 2$.
Describe the movement of the point P at the instants when

4. $\alpha = \pi/3$

5. $x = 1.5$

6. $\cos \alpha = 2/3$.

Susan.

Rates of change

1. In ΔAOF and ΔABP

$\angle BAP$ is common

$\angle AOF = \angle OPB = 90°$ (perpendicular to the ground)

$\therefore \angle AFO = \angle FBP$ ($\angle$ sum of Δ)

$\therefore \Delta AOF \,|||\, \Delta ABP$

$\therefore \tan\alpha = \frac{x}{h}$ ✓

$\therefore x = \tan\alpha \, h$ ✓

2. $AP \Rightarrow \sin\alpha = \frac{AP}{l}$

$AP = \sin\alpha \, l$ ✓

$\therefore y = AP - x$

which is $y = l\sin\alpha - \tan\alpha \, h$ ✓

3. $y = l\sin\alpha - h\tan\alpha$

$\frac{dy}{d\alpha} = l\cos\alpha - h\sec^2\alpha$

$\frac{dx}{d\alpha} = h\sec^2\alpha$

$\frac{dy}{dx} = \frac{dy}{d\alpha} \times \frac{dx}{d\alpha}$

$= \frac{dy}{d\alpha} \times \frac{d\alpha}{dx}$ ✓

$= (l\cos\alpha - h\sec^2\alpha) \times \frac{1}{h\sec^2\alpha}$

$= \left[l\cos\alpha - \frac{h}{\cos^2\alpha}\right] \times \frac{1}{\frac{h}{\cos^2\alpha}} = \left[l\cos\alpha - \frac{h}{\cos^2\alpha}\right] \times \frac{\cos^2\alpha}{h}$ ✓

$= \frac{l\cos^3\alpha - h}{\cancel{\cos^2\alpha}} \times \frac{\cancel{\cos^2\alpha}}{h}$

$= \frac{l\cos^3\alpha - h}{h}$ ✓

$= \frac{l}{h}\cos^3\alpha - 1$ as required.

4. $\frac{dx}{dt} = 8$ $\qquad \frac{dy}{d\alpha} \times \frac{dx}{d\alpha} \times \frac{dy}{dt}$

$$\frac{dx}{dt} = \frac{dy}{d\alpha} \times \frac{dx}{d\alpha} \times \frac{dy}{dt} \quad ✓$$

$$\frac{dx}{dt} = \frac{dy}{d\alpha} \times \frac{d\alpha}{dx} \times \frac{dy}{dt}$$

$$8 = (\cos\alpha - h\sec^2\alpha) \times \frac{1}{h\sec^2\alpha} \, \frac{dy}{dt}$$

where $\alpha = \frac{\pi}{3}$

$$8 = \left(\cos\frac{\pi}{3} - 2\frac{1}{\cos^2\frac{\pi}{3}}\right) \times \frac{1}{2\frac{1}{\cos^2\frac{\pi}{3}}} \, \frac{dy}{dt}$$

$$8 = (0.5 - 8) \times \frac{1}{8}$$

$$8 = -0.9375$$

$$\frac{dx}{dt} = \frac{d\alpha}{dy} \times \frac{dx}{d\alpha} \times \frac{dy}{dt} \quad ✓$$

$$8 = \frac{1}{(\cos\alpha - h\sec^2\alpha} \times h\sec^2\alpha \times \frac{dy}{dt}$$

$$= \frac{1}{6.75\times\cos\frac{\pi}{3} - 2\frac{1}{\cos^2\frac{\pi}{3}}} \times \frac{2}{\cos^2\frac{\pi}{3}} ✓ \times \frac{dy}{dt}$$

$$8 = \frac{1}{3.375 - 8} \times 8 \times \frac{dy}{dt}$$

$$1 = \frac{1}{3.375 - 8}$$

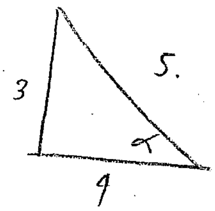
$$\frac{dy}{dt} = -4.625 \text{ m/min} \quad ✓$$

5. $\frac{dx}{dt} = \frac{d\alpha}{dy} \times \frac{dx}{d\alpha} \times \frac{dy}{dt}$

$x = \tan\alpha\, h.$

$1.5 = \tan\alpha \times 2.$

$\tan\alpha = 3/4.$ ✓



$\therefore \cos\alpha = \frac{4}{5}.$ ✓

$\frac{dx}{dt} = \frac{1}{l\cos\alpha - h\sec^2\alpha} \times h\sec^2\alpha \times \frac{dy}{dt}$

$8 = \frac{1}{6.75 \times \frac{4}{5} - 2\frac{1}{(\frac{4}{5})^2}} \times 2 \times \frac{1}{(\frac{4}{5})^2} \times \frac{dy}{dt}$ ✓

$8 = \frac{1}{5.4 - 3.125} \times 3.125 \frac{dy}{dt}$ ✓

$= \frac{3.125}{2.275}$

$= 1.3736$ ✓

$\therefore 6.62637\ m/min. = \frac{dy}{dt}$

6. $\frac{dx}{dt} = \frac{d\alpha}{dy} \times \frac{dx}{d\alpha} \times \frac{dy}{dt}$

$8 = \frac{1}{l\cos\alpha - h\sec^2\alpha} \times h\sec^2\alpha \frac{dy}{dt}$

$= \frac{1}{6.75 \times 2/3 - \frac{2}{\frac{4}{9}}} \times 2 \times \frac{1}{\frac{4}{9}} \times \frac{dy}{dt}$

$8 = \frac{dy}{dt}.$ ✓

$\therefore 8\ m/min.$