RANDWICK GIRLS' HIGH SCHOOL

Facta Non Verba

Mathematics Department

Year 12

Extension I Mathematics

Assessment Task

June 2006

Instructions to Candidates:

- Approved Scientific Calculators may used.
- Show <u>ALL</u> necessary working.
- Answer questions on paper provided

Time Allowed: 50 minutes

Question	Marks
1	/10
2	/9
3	/10
4	/8
5	/6
Total	/43

Question 1.

a) Find the derivative of $\sin^{-1}(2x)$ **3**

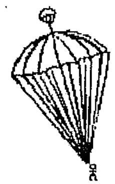
b) Using the table of Standard Integrals show that

$$\int_0^1 \frac{5}{\sqrt{2-x^2}}\,dx = \frac{5\pi}{4}$$ **3**

c) **i)** State the domain and range of $\cos^{-1}(x)$ **1**

ii) Sketch $3\cos^{-1}\left(\frac{x}{2}\right)$ *for the most appropriate domain* **3**

Question 2.



A sky-diver opens her parachute when falling at $30\ ms^{-1}$.
Thereafter her acceleration is given by $\frac{dv}{dt} = k(6-v)$, where k is a constant.

a) Show that this condition is satisfied when $v = 6 + Ae^{-kt}$ **2**

b) Find the value of the constant, A. **1**

c) One second after opening the parachute, her velocity has decreased to $10.7\ ms^{-1}$.
Find the value of k correct to 2 decimal places. **3**

d) Find the velocity, correct to 1 decimal places, 2 seconds after the parachute is opened. **2**

e) What is the minimum velocity of the sky-diver at the time of landing? **1**

Question 3.

The position of a particle moving along the x axis is given by
$x = 2\cos\left(3t + \frac{\pi}{6}\right)$ where x is measured in metres.

a) State the amplitude of the motion. **1**

b) State the period of the motion. **1**

c) Find the velocity of the particle as a function of t. **1**

d) Show that the acceleration of the particle is given by

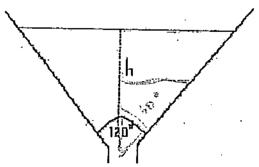
$$\ddot{x} = -n^2 x$$ **2**

and find the value of n

e) What was the particle's initial position and in what direction was it moving? **2**

f) What time elapsed before the particle was next at its initial position and what was its velocity at that time. **3**

Question 4.



A filter funnel with a vertical angle of 120° contains liquid to a depth of h cm.

a) Show that the volume of the liquid in the filter funnel is given by

$$V = \pi h^3$$ **3**

b) Show that the horizontal surface area of the liquid in the funnel when viewed from above is given by $SA = 3\pi h^2$ **2**

c) If the volume is decreasing at a rate of 30 mL / minute, at what rate is the height decreasing when the surface area is $40\ cm^2$? **3**

Question 5. (6 marks)

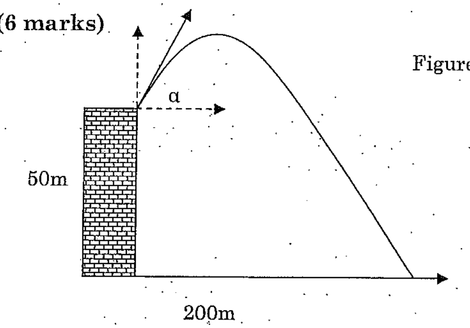


Figure not to scale.

The diagram shows the path of a projectile launched at an angle of elevation α, with an initial velocity of 40 m/s, from the top of a 50 metre high building. The acceleration due to gravity is assumed to be 10 m/s².

i) Given that $\frac{d^2x}{dt^2} = 0$ *and* $\frac{d^2y}{dt^2} = -10$

show that $x = 40t\cos\alpha$ *and* $y = -5t^2 + 40t\sin\alpha + 50$, where x and y are horizontal and vertical displacements of the projectile in metres from O at time t seconds after launching.

ii) The projectile lands on the ground 200 metres from the base of the building. Find two possible values of α. Give you answers to the nearest degree.

Mrs P.

H-L Lim

Q1.

a) $\sin^{-1} 2x$.

$f'(x) = \frac{1}{\sqrt{1-4x^2}} \times 2$

$= \frac{2}{\sqrt{1-4x^2}}$

3

b) $\int_0^1 \frac{5}{\sqrt{2-x^2}}\,dx$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}$

$= 5\int_0^1 \frac{1}{\sqrt{(\sqrt{2})^2-x^2}}\,dx$

$= 5\left(\sin^{-1}\frac{x}{\sqrt{2}}\right)_0^1$

$= 5\left(\sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1} 0\right)$

$= \frac{5\pi}{4}$.

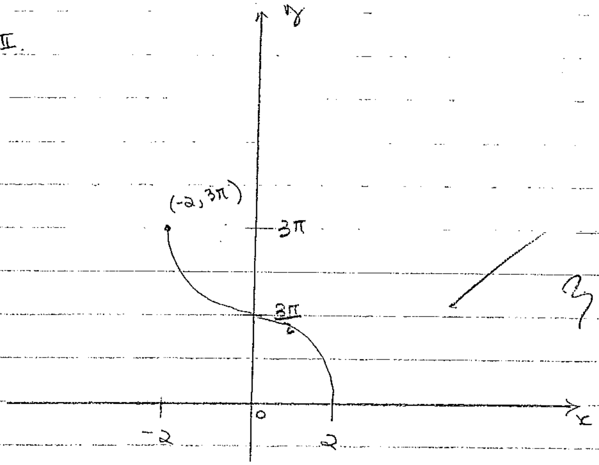
3.

c) i $\cos^{-1} x$.

D: $-1 \le x \le 1$

R: $0 \le y \le \pi$

ii.



D: $-1 \le \frac{x}{2} \le 1$

$-2 \le x \le 2$

R: $0 \le \frac{y}{3} \le \pi$

$0 \le y \le 3\pi$.

3

Mrs P

H-L Lim.

Q2.

a) $V = 6 + Ae^{-kt} \Rightarrow V - 6 = Ae^{-kt}$

$\frac{dV}{dt} = Ae^{-kt} \times -k$.

$= -k(V-6)$

$= k(6-V)$

b) $30 = 6 + Ae^{-k\cdot(0)}$ $t=0$

$30 = 6 + A$

~~A = 24 m/s⁻¹~~

$A = 24$.

c) $t = 1$.

$V = 10.7$ m/s

$k = ?$

$10.7 = 6 + 24e^{-k\cdot 1}$

$\frac{10.7-6}{24} = e^{-k}$

$\frac{47}{240} = e^{-k}$

$\ln\frac{47}{240} = \ln e^{-k}$

$-k = \ln\frac{47}{240}$

$k = -\ln\frac{47}{240}$

$= 1.63$. (2.dp).

d) $t = 2$. $V = 6 + 24e^{-1.63\cdot(2)}$

$= 6 + 24e^{-3.26}$

$= 6.92$ m/s.

$= 6.9$ m/s (1.dp.).

e) ~~Minim~~ minimum occurs when $\frac{dV}{dt} = 0$ & $\frac{d^2V}{dt^2} > 0$.

$0 = 1.63(6-V)$ time of land is when $V=0$

$= 1.63(6-0)$

$= 9.78$ m/s.

Q2 continued.

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H.L Lim

$k(6-V) = \frac{dV}{dt}$.

$\frac{d^2V}{dt^2} = 6-k$.

$= 6 - 1.63 > 0 \quad \therefore$ it's a Min.

Q3.

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$x = 2\cos(3t + \frac{\pi}{6})$.

a) amplitude $= 2$.

b) period $= \frac{2\pi}{n} = \frac{2\pi}{3}$ s.

c) $\frac{dx}{dt} = v = -2\sin(3t + \frac{\pi}{6}) \times 3$

$= -6\sin(3t + \pi/6)$

d) $a = \ddot{x} = \frac{d^2x}{dt^2} = -6\cos(3t + \pi/6) \times 3$

$= -18\cos(3t + \pi/6)$

$= -9 \cdot 2\cos(3t + \pi/6)$

$= -9x$.

$= -3^2 x$.

$= -n^2 x$.

$n = 3$.

e) $t = 0$. $x = ?$

$x = 2\cos(3(0) + \pi/6)$

$= \frac{\cancel{2}\sqrt{3}}{\cancel{2}}$

$= \sqrt{3}$ m.

$\frac{dx}{dt} = -6\sin(3t + \pi/6)$

$= -6\sin(0 + \pi/6)$

$= \frac{-6}{2}$

$= -3$ m/s

The particle was originally at $x = \sqrt{3}$ & moving to the left (negative direction).

f) $x = 2\cos(3t + \pi/6)$

$\frac{\sqrt{3}}{2} = \cos(3t + \pi/6)$

$\cos(3t + \pi/6) = \frac{\pi}{6}$

$3t + \pi/6 = \frac{\pi}{6}, \frac{11\pi}{6} \ldots$ after $t = 0$.

$3t = \frac{11\pi}{6} - \frac{\pi}{6}$

Q3 continued.

$3t = \frac{5\pi}{3}$

$= \frac{5\pi}{3} \times \frac{1}{3}$

$t = \frac{5\pi}{9}$

$V = -6\sin(3t + \pi/6)$

$= -6\sin\left(\frac{\cancel{3}\pi\cdot 5}{\cancel{9}3} + \pi/6\right)$

$= -6\sin\left(\frac{11\pi}{6}\right)$

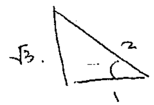
$= -6 \cdot -\frac{1}{2}$

at $t = \frac{5\pi}{9}$, $V = 3$ m/s.

Q4

a) ~~By similar Δ's~~

relationship between r & h:



$\tan 60° = \frac{r}{h}$

$h\tan 60° = r$

$\sqrt{3}h = r$

filter funnel $= \frac{1}{3}\pi r^2 h$.

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(\sqrt{3}h)^2 h$

$V = \pi h^3$

3

b) πr^2 = surface area

$SA = \pi r^2$

$= \pi(\sqrt{3}h)^2$

$= 3\pi h^2$

2

t = min

V = mL.

c) $\frac{dV}{dt} = 30$ ml/min.

$\frac{dh}{dt} = ?$

$SA = 40 = 3\pi h^2$

$\frac{dV}{dh} = 3\pi h^2$

$\frac{40}{3} = \pi h^2$.

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$\frac{dh}{dt} = \frac{1}{3\pi h^2} \times 30$

$= \frac{10}{\pi h^2}$ using $\frac{40}{3} = \pi h^2$.

$= \frac{10}{40/3}$

$= 0.75$ cm/min

3

8/8

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H.L Lim

Q5.

i $\frac{d^2x}{dt^2} = 0.$ $\frac{d^2y}{dt^2} = -10.$

Initial conditions: $t=0$ $x=0$ $y=50$ $\frac{dx}{dt} = v\cos\alpha = 40\cos\alpha$ $\frac{dy}{dt} = v\sin\alpha = 40\sin\alpha.$

$\frac{dx}{dt} = 40\cos\alpha.$ $\frac{dy}{dt} = -10t + C_2$

$x = 40\cos\alpha t + c_1$ $40\sin\alpha = 0 + C$

$0 = 0 + C$ $C = 40\sin\alpha.$

$C = 0$ $\frac{dy}{dt} = -10t + 40\sin\alpha.$

$x = 40t\cos\alpha.$

$y = \frac{-10t^2}{2} + 40t\sin\alpha + C_3.$

$50 = -5(0)^2 + 40(0)\sin\alpha + C_3$

$C_3 = 50$

$y = -5t^2 + 40t\sin\alpha + 50.$

(II) Range is when $y=0$. $x=200$.

cartesian equation:

$\frac{x}{40\cos\alpha} = t.$ $y = -5\left(\frac{x^2}{40^2\cos^2\alpha}\right) + \frac{40x}{40\cos\alpha}\sin\alpha + 50$

$y = \frac{-x^2}{320\cos^2\alpha} + x\tan\alpha + 50$

~~$0 = \frac{-x^2}{320}(1+\tan^2\alpha) + x\tan\alpha + 50$~~

~~$= \frac{-x^2}{320} - x^2$~~

6

Q5 continued.

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$(200, 0)$: $y = \frac{-x^2}{320}(1+\tan^2\alpha) + x\tan\alpha + 50.$

$0 = \frac{-200^2}{320}(1+\tan^2\alpha) + 200\tan\alpha + 50.$

$= -125 - 125\tan^2\alpha + 200\tan\alpha + 50.$

$0 = \frac{125\tan^2\alpha - 200\tan\alpha + 75}{25}$

$= 5\tan^2\alpha - 8\tan\alpha + 3.$

$\tan\alpha = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{+8 \pm \sqrt{64 - 4\times5\times3}}{10}$

$= \frac{8 \pm 2}{10}$

$= 1$ or $0.6.$

$\tan\alpha = 1$ or $\tan\alpha = 0.6$

$\alpha = \frac{\pi}{4}$ $\alpha = 30°58'.$

$= 45°.$