SYDNEY GIRLS HIGH SCHOOL

LABOR OMNIA VINCIT

2006 HSC Assessment Task 3

June 13, 2006

MATHEMATICS Extension 1

Year 12

Time allowed: 80 minutes

Topics: Parametrics, Circle Geometry, Inverse Functions, Integration by Substitution

DIRECTIONS TO CANDIDATES:

- Attempt all questions
- Questions are of equal value
- There are 4 questions with part marks shown in brackets
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work
- Board approved calculators may be used
- Write on one side of the paper only

QUESTION 1

MARKS

a) Find the inverse of the following functions and state the domain and range

(i) $y = \log_e(x-3)$ — 2

(ii) $y = x^2 - 4x + 5 \quad x \geq 2$ — 2

b) Differentiate

(i) $y = \sin^{-1} 3x$ — 2

(ii) $y = \cos^{-1} \dfrac{x}{4}$ — 2

c) Find the primitive function of

(i) $\displaystyle\int \frac{1}{4+x^2}\,dx$ — 1

d) $f(x) = x\sin^{-1} x$

(i) what is the domain of f(x) — 1

(ii) show that this is an even function — 2

(iii) verify that when x = 0, f(x) is stationary — 2

(iv) sketch a graph of y = f(x) — 1

e) (i) Show that $\dfrac{d(x^2 \tan^{-1} x)}{dx} = 2x\tan^{-1} x + 1 - \dfrac{1}{1+x^2}$ — 2

(ii) Hence or otherwise find $\displaystyle\int_0^{\sqrt{3}} x\tan^{-1} x\,dx$ — 3

QUESTION 2

a) Find the following indefinite integrals using the substitution given — MARKS

(i) $\int x\sqrt{x^2+4}\,dx$ $\qquad u = x^2 + 4$ — 2

(ii) $\int \frac{dx}{x(\log x_e)^3}$ $\qquad u = \log x_e$ — 2

(iii) $\int \frac{e^x dx}{\sqrt{49 - e^x}}$ $\qquad u = e^x$ — 2

b) Evaluate the following definite integrals using the substitution given

(i) $\int_{-5}^{0} \frac{t\,dt}{\sqrt{4-t}}$ $\qquad t = 4 - u^2$ — 4

(ii) $\int_{0}^{\frac{\pi}{2}} \frac{\sin\theta}{3 - 2\cos\theta}\,d\theta$ $\qquad y = 3 - 2\cos\theta$ — 4

c) The region R is bounded by the curve $y = \frac{x}{x+1}$ the x-axis and the vertical line x = 3.

Use the substitution u = x + 1 to find

(i) the exact area R — 3

(ii) the exact volume generated when R is rotated about the x-axis — 3

QUESTION 3 — MARKS

a) T $(2at, at^2)$ is a variable point on the parabola $x^2 = 4ay$ — 3

(i) show that the gradient of the tangent at T is t.

(ii) show that the equation of the tangent at T is $y = tx - at^2$

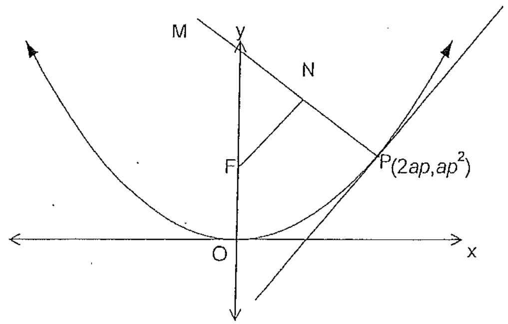
b) Write down the equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the parabola $x^2 = 4ay$ — 4

(i) find the equation of the chord of contact from the point (3,-2) to the parabola $x^2 = 8y$

(ii) at what point does the line intersect the directrix

c) If PM is a normal to the parabola $x^2 = 4ay$ at a variable point P$(2ap, ap^2)$ and FN is drawn through the focus F parallel to the tangent at P to cut the normal at N — 4

(i) prove that the locus of N (x, y) is $x^2 = a(y - a)$



d) The tangents at $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ on the parabola $x^2 = 4ay$ intersect at T (a(p+q), apq)

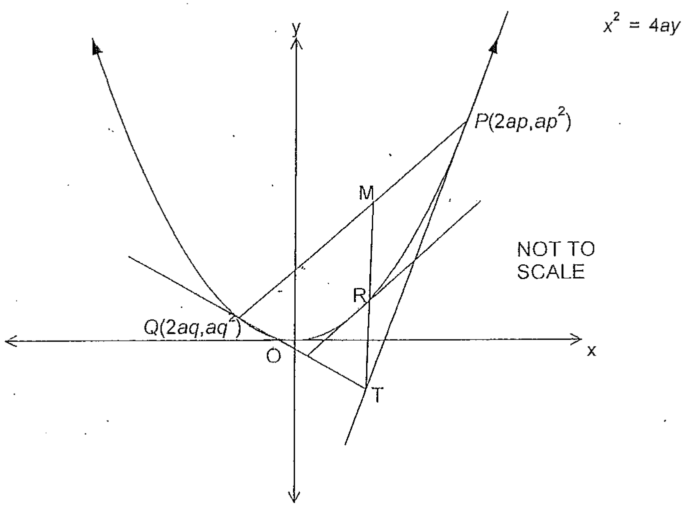
(i) find M the midpoint of PQ

Hence show that

(ii) TM is parallel to the axis of symmetry

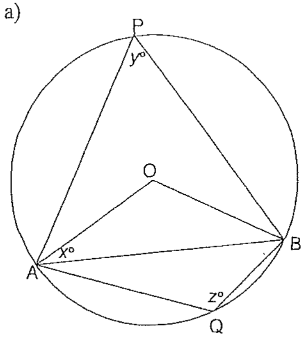
(iii) if TM meets the parabola on R, then R bisects TM

(iv) the tangent at R is parallel to the chord PQ



QUESTION 4 | MARKS

a)

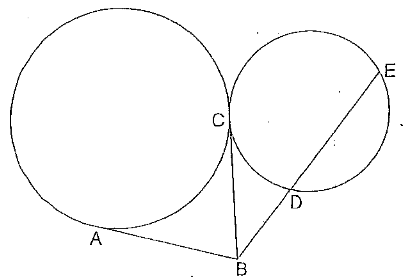


O is the centre of the circle 5
Prove that

(i) $x + y = 90$

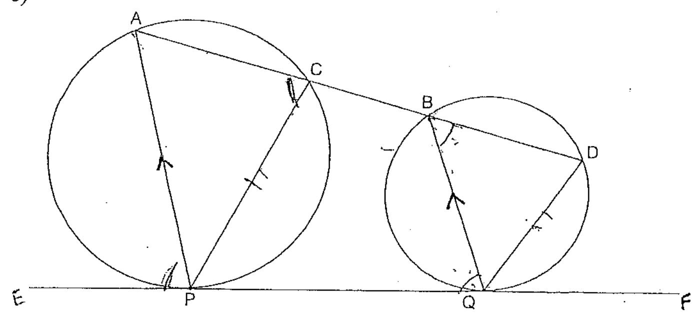
(ii) $z - y = 2x$

b) 3



BA and BC are tangents to the circles
$DE = 5 \times BD$. Prove $BA = \sqrt{6} \times BD$

c) 5

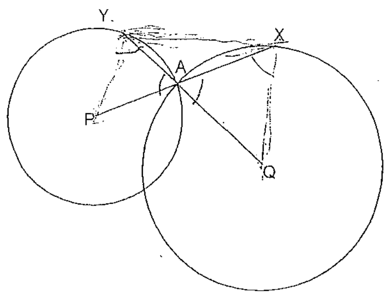


PQ is a common tangent and $PA \parallel QB$. Prove that

(i) $PC \parallel QD$

(ii) PQBC is a cyclic quadrilateral

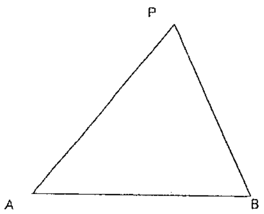
d) 4



P and Q are the centres of the circles
PAX and QAY are straight lines.

Prove that P, Q, X and Y are concyclic

e) 3



A and B are fixed points. P moves on the plane so that AB subtends an angle of 30^0 at P.

(i) describe the locus of P

(ii) describe what construction you would carry out to draw the locus of P

THE END

Yr 12 Ext 2, June 06

a) i) if $y = \log_e(x-3)$

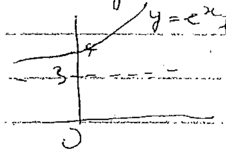
$f^{-1}: x = \log_e(y-3)$

$y-3 = e^x$

$y = e^x + 3$

Domain = the set of reals

Range = $\{y : y > 3\}$



(2)

ii) if $y = x^2-4x+5,\ x \geq 2$

$f^{-1}: x = y^2-4y+5,\ y \geq 2$

$x-5 = y^2-4y$

$x-5+4 = y^2-4y+4$

$x-1 = (y-2)^2,\ y \geq 2$

$y-2 = \sqrt{x-1},\ y \geq 2$

$y = 2+\sqrt{x-1},\ y \geq 2$

Domain = $\{x : x \geq 1\}$

Range = $\{y : y \geq 2\}$ (2)

i) $y = \sin^{-1}3x$

$y' = \frac{1}{\sqrt{1-(3x)^2}} \times 3$

$= \frac{3}{\sqrt{1-9x^2}}$ (2)

ii) $y = \cos^{-1}\frac{x}{4}$

$y' = \frac{-1}{\sqrt{16-x^2}}$ (2)

i) $\int \frac{1}{4+x^2}\,dx$

$= \frac{1}{2}\tan^{-1}\frac{x}{2} + C$ (1)

d) i) $f(x) = x\cdot\sin^{-1}x$

Domain = $\{x : -1 \leq x \leq 1\}$ (1)

ii) $f(a) = a\cdot\sin^{-1}a$

$f(-a) = -a \times \sin^{-1}(-a)$

$= -a \times -\sin^{-1}(a)$

$= a \times \sin^{-1}(a)$ (2)

$\therefore f(-a) = f(a)$

$\therefore f(x)$ is an even function

iii) $f(x) = x\cdot\sin^{-1}x$

$f'(x) = v\frac{du}{dx} + u\cdot\frac{dv}{dx}$

$= \sin^{-1}x \times 1 + x \times \frac{1}{\sqrt{1-x^2}}$

$= \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$

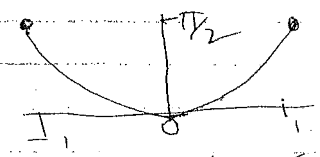
$f'(0) = \sin^{-1}0 + \frac{0}{\sqrt{1-0}}$

$= 0 + 0$ (2)

$\therefore f'(0) = 0$

$\therefore$ When $x=0$, $(x, f(x))$ is stationary.

iv)



(1)

e) i) $\frac{d}{dx}(x^2\cdot\tan^{-1}x) = v\frac{du}{dx} + u\frac{dv}{dx}$

$= \tan^{-1}x \times 2x + x^2 \times \frac{1}{1+x^2}$

$\therefore \frac{d}{dx}(x^2\cdot\tan^{-1}x) = 2x\cdot\tan^{-1}x + \frac{x^2}{x^2+1}$

$= 2x\cdot\tan^{-1}x + \frac{x^2+1-1}{x^2+1}$

$\therefore \frac{d}{dx}(x^2\cdot\tan^{-1}x) = 2x\cdot\tan^{-1}x + \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}$

$\therefore \frac{d}{dx}(x^2\cdot\tan^{-1}x) = 2x\cdot\tan^{-1}x + 1 - \frac{1}{x^2+1}$

(3)

(ii) $\frac{d}{dx}\left(x^2 \tan^{-1}x\right) = 2x \cdot \tan^{-1}x + 1 - \frac{1}{x^2+1}$

$x^2 \cdot \tan^{-1}x = \int 2x \cdot \tan^{-1}x\,dx + \int 1\,dx - \int \frac{dx}{x^2+1}$

$x^2 \cdot \tan^{-1}x = \int 2x \cdot \tan^{-1}x\,dx + (x - \tan^{-1}x + c)$

$x^2 \tan^{-1}x - x + \tan^{-1}x + c = \int 2x \tan^{-1}x\,dx$

$\int_0^{\sqrt{3}} 2x \tan^{-1}x\,dx = \left[x^2 \tan^{-1}x - x + \tan^{-1}x + c\right]_0^{\sqrt{3}}$

$\int_0^{\sqrt{3}} x \tan^{-1}x\,dx = \left[\frac{x^2}{2} \tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + c\right]_0^{\sqrt{3}}$

$= \left(\frac{3}{2}\tan^{-1}\sqrt{3} - \frac{\sqrt{3}}{2} + \frac{1}{2}\tan^{-1}\sqrt{3} + c\right) - (0 - 0 + 0 + c)$

$= \frac{3}{2} \times \frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{\pi}{3}$

$= \frac{4}{6}\pi - \frac{\sqrt{3}}{2}$

$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

(3)

c)(i) $\int x\sqrt{x^2+4}\,dx$, $u = x^2+4$, $du = 2x\,dx$, $\frac{du}{2} = x\,dx$

$= \int \cancel{\sqrt{x^2+4} \times x\,dx}$

$= \int \sqrt{u} \times \frac{1}{2}\,du$

$= \int \frac{1}{2} u^{1/2}\,du$

$= \frac{1}{2} \times \frac{2}{3} u^{3/2} + c$

$= \frac{1}{3}(x^2+4)^{3/2} + C$ (2)

(ii) $\int \frac{dx}{x(\log_e x)^3}$, $u = \log_e x$, $du = \frac{1}{x}\,dx$

$= \int \frac{du}{u^3}$

$= \int u^{-3}\,du$

$= -\frac{1}{2}u^{-2} + c$

$= -\frac{1}{2u^2} + c$

$= -\frac{1}{2(\log_e x)^2} + c$ (2)

(iii) $\int \frac{e^x\,dx}{\sqrt{49-e^x}}$, $u = e^x$, $du = e^x\,dx$

$= \int \frac{du}{\sqrt{49-u}}$

$= \int (49-u)^{-1/2}\,du$

$= -2(49-u)^{1/2} + c$

$= -2\sqrt{49-e^x} + c$ (2)

b)(i) $\int_{-5}^{0} \frac{t\,dt}{\sqrt{4-t}}$, $t = 4-u^2$, $dt = -2u\,du$

$= \int_3^2 \frac{(4-u^2)(-2u)\,du}{\sqrt{4-(4-u^2)}}$ $t=0$, $0 = 4-u^2$, $u=2$

$= \int_3^2 \frac{-2u(4-u^2)\,du}{\sqrt{4-4+u^2}}$ $t=-5$, $-5 = 4-u^2$, $u^2 = 9$, $u = 3$

$= \int_2^3 \frac{2u(4-u^2)\,du}{\sqrt{u^2}}$

$= \int_2^3 \frac{2u(4-u^2)\,du}{u}$

$= \int_2^3 2(4-u^2)\,du$

$= \int_2^3 (8-2u^2)\,du$

$= \left[8u - \frac{2}{3}u^3 + c\right]_2^3$

$= \left(8\times 3 - \frac{2}{3}\times 3^3 + c\right) - \left(8\times 2 - \frac{2}{3}\times 2^3 + c\right)$

$= (24 - 18) - (16 - 16/3)$

$= 6 - 16 + 16/3$

$= -10 + 5\tfrac{1}{3}$

$= -4\tfrac{2}{3}$. (4)

b)(ii) $\int_0^{\pi/2} \frac{\sin\theta}{3-2\cos\theta}\,d\theta$, $y = 3-2\cos\theta$, $dy = 2\sin\theta\,d\theta$

$\theta = \pi/2$, $y = 3 - \cos\frac{\pi}{2} = 3 - 0 = 3$

$\theta = 0$, $y = 3 - 2\times\cos 0 = 3 - 2 = 1$

$= \int_1^3 \frac{\frac{1}{2}\,dy}{y}$

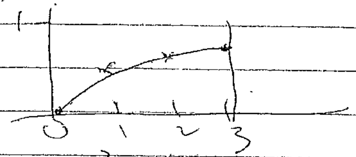
$= \int_1^3 \frac{dy}{2y}$

$= \left[\frac{1}{2}\ln y + c\right]_1^3$

$= \left(\tfrac{1}{2}\ln 3 + c\right) - \left(\tfrac{1}{2}\ln 1 + c\right)$ (4)

$= \tfrac{1}{2}\ln 3$.

c)(i) $y = \frac{x}{x+1}$



$A = \int_0^3 \frac{x}{x+1}\,dx$, $u = x+1$, $du = 1\,dx$

$= \int_1^4 \frac{u-1}{u}\,du$

$= \int_1^4 1 - \frac{1}{u}\,du$

$= \left[u - \ln u + c\right]_1^4$

$= (4 - \ln 4 + c) - (1 - \ln 1 + c)$

$= 4 - \ln 4 - 1 + \ln 1$

$= (3 - \ln 4)$ sq units (3)

(ii) $V = \pi\int_0^3 y^2\,dx$

$= \pi\int_0^3 \frac{x^2}{(x+1)^2}\,dx$

$= \pi\int_1^4 \frac{(u-1)^2}{u^2}\,du$

$= \pi\int_1^4 \frac{u^2-2u+1}{u^2}\,du$

$= \pi\int_1^4 1 - \frac{2}{u} + \frac{1}{u^2}\,du$

$= \pi\int_1^4 1 - \frac{2}{u} + u^{-2}\,du$

$= \pi\left[u - 2\ln u - u^{-1} + c\right]_1^4$

$= \pi\left[\left(4 - 2\ln 4 - \frac{1}{4}\right) - \left(1 - 2\ln 1 - \frac{1}{1}\right)\right]$

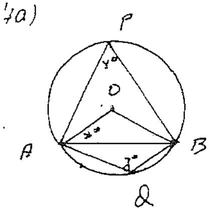
$= \pi\left(3\tfrac{3}{4} - 2\ln 4 + 2\ln 1\right)$

$= \pi\left(\frac{15}{4} - 2\ln 4\right)$

cu units

(3)

4a)



i) $\angle AOB = 2y°$ (angle at centre) 1

$\angle OBA = x°$ (isos Δ, $AO = OB$) 1

In ΔAOB

$x + x + 2y = 180$ (angle sum of Δ)

$\therefore 2x + 2y = 180$ 1

$x + y = 90$

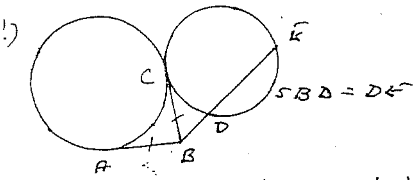
ii) $y + z = 180$ (opp $\angle$'s of cyclic quad)

& $2x + 2y = 180$ (from above) 1

$\therefore y + z = 2x + 2y$

$\therefore z - y = 2x$. 1

b)



$5BD = DE$

1

$AB = BC$ (tangents to a circle) 1

$BC^2 = BD \cdot BE$ (tangent/intercept thm)

Let $BD = x$, $\therefore DE = 5x$, $BE = 6x$.

$\therefore BC^2 = x \cdot 6x$

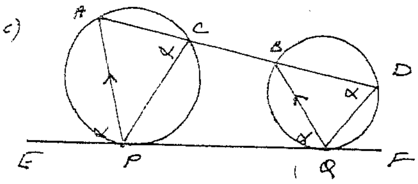
$= 6x^2$

But $AB = BC$

$\therefore AB^2 = 6x^2$ 1

$AB = \sqrt{6} \cdot x = \sqrt{6} \cdot BD$

4c)



i) Let $\angle APE = \alpha$

$\therefore \angle BQE = \alpha$ (corres $\angle$'s $AP \parallel BQ$) 1

& $\angle$'s $ACP = \angle BDQ = \alpha$ (angle in alt. segment) 1

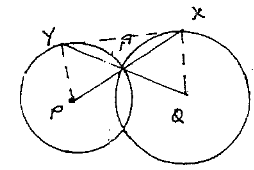
$\therefore PC \parallel DQ$ (corres. $\angle$'s C & D are equal) 1

ii) If $\angle ACP = \alpha$

$\angle PCB = 180 - \alpha$ (st. angle) 1

$\therefore PCBQ$ is a cyclic quad (opp $\angle$'s are supplementary) 1

d)



Let $\angle PYA = \alpha$ 1

$\therefore \angle YAP = \alpha$ (isos Δ, PY, PA radii)

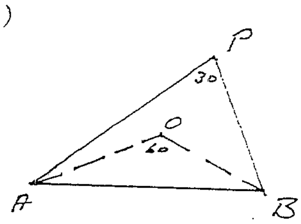
$\therefore \angle XAQ = \alpha$ (vert opp $\angle$'s) 1

$\therefore \angle AXQ = \alpha$ (isos Δ, AQ, XQ radii)

$= \angle PYA$ 1

$\therefore PYXQ$ is a cyclic quad because P & Q are subtending equal angles. 1

e)



1

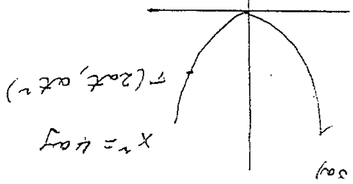
i) P is the major arc of a circle.

ii) If the angle at P on the circumference is 30°, the angle at the centre is 60°. 2

$\therefore$ Construct 60° angles at A & B & the centre of the circle is where the construction lines me[et]

$\therefore$ With compass on point O & radius OA. draw the major arc of a circle.

5a)



$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$

at $x = 2at$, $\frac{dy}{dx} = \frac{2at}{2a} = t$ 1

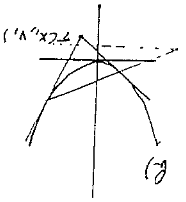
$\therefore T$ is $y - y_1 = m(x - x_1)$

$\therefore y - at^2 = t(x - 2at)$

$y - at^2 = tx - 2at^2$

$y = tx - at^2$ 2

b)



$x_1 x = 2a(y + y_1)$ 1

$x_1 = 3$, $y_1 = -2$, $a = 2$

$\therefore 3x = 4(y - 2)$

$3x = 4y - 8$

$3x - 4y + 8 = 0$ 2

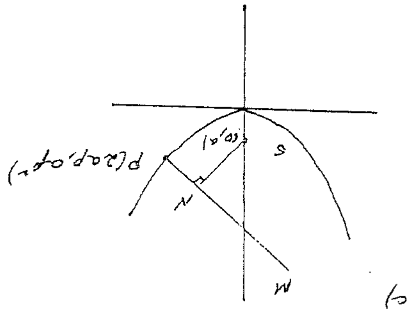
at $y = -2$, $3x + 8 + 8 = 0$

$3x = -16$

$x = -16/3$

$\therefore Pt$ is $(-16/3, -2)$ 1

c)



$x^2 = 4ay$

$2x = 4a \cdot \frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{x}{2a}$

at $x = 2ap$, $\frac{dy}{dx} = p$

$\therefore m_N = -\frac{1}{p}$

& Eqn is $y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$ 1

$\therefore x + py = 2ap + ap^3$

Line through focus is

$y - a = p(x - 0)$ 1

$y = px + a$.

Sub into normal.

$\therefore x + p(px + a) = 2ap + ap^3$

$x + p^2 x = ap + ap^3$

$x(1 + p^2) = ap(1 + p^2)$ 1

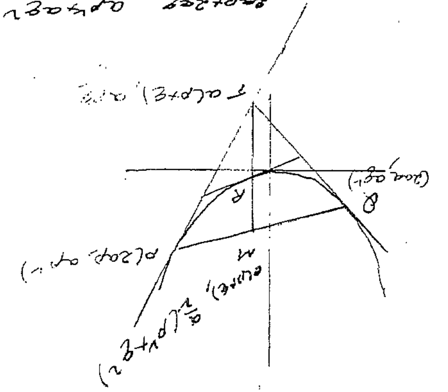
$\therefore x = ap$ & $y = ap^2 + a$.

$\therefore \frac{x}{a} = p \Rightarrow y = a(\frac{x^2}{a^2}) + a$

$y = \frac{x^2}{a} + a$

$ay = x^2 + a^2$ 1

$\therefore x^2 = a(y - a)$



i) M is $\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}$ 1

$= a(p+q), \frac{a}{2}(p^2 + q^2)$ 1

ii) M & T have the same x value,

$\therefore$ line is vertical (i.e. parallel to axis)

iii) Midpoint MT is

$a(p+q), \frac{1}{2}\{\frac{a}{2}(p^2+q^2) + apq\}$

$= a(p+q), \frac{1}{2}\{\frac{ap^2 + aq^2 + 2apq}{2}\}$

$= a(p+q), \frac{a}{4}(p+q)^2$.

Sub into $x^2 = 4ay$

$LHS = a^2(p+q)^2$

$RHS = 4a \cdot \frac{a}{4}(p+q)^2 = a^2(p+q)^2 = L[HS]$

$\therefore R$ lies on $x^2 = 4ay$. 2

iv) $\frac{dy}{dx} = \frac{x}{2a}$. At R, $\frac{dy}{dx} = \frac{1}{2a} \cdot a(p+q)$

$= \frac{p+q}{2}$

$m_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq} = \frac{a(p-q)(p+q)}{2a(p-q)} = \frac{p+q}{2}$

$=$ grad at