South Sydney High School
INTEGRATION BY SUBSTITUTION
3 Unit Worksheet

INDEFINITE INTEGRALS

Find the following integrals, using the indicated change of variable.

(a) $\int x(x^2 + 4)^3 \, dx$

$u = x^2 + 4$

(b) $\int \frac{x^3}{\sqrt{x^4 + 4}} \, dx$

$u = x^4 + 4$

(c) $\int x(2 - x)^3 \, dx$

$u = 2 - x$

(d) $\int x\sqrt{x + 2} \, dx$

$u = x + 2$

(e) $\int \frac{1}{9 + (2x+1)^2} \, dx$

$u = 2x + 1$

(f) $\int \frac{1}{x \ln x} \, dx$

$u = \ln x$

(g) $\int \sin^4 x \cos x \, dx$

$u = \sin x$

(h) $\int \sec^2 x \tan x \, dx$

$u = \tan x$

(i) $\int x \, e^{x^2} \, dx$

$u = x^2$

(j) $\int \frac{1}{\sqrt{x}(1 + x)} \, dx$

$u = \sqrt{x}$

(k) $\int \frac{1}{\sqrt{4 - (3x-1)^2}} \, dx$

$u = 3x - 1$

(l) $\int \frac{x}{(2x + 1)^3} \, dx$

$u = 2x + 1$

DEFINITE INTEGRALS

Evaluate the following, using the indicated change of variable.

(a) $\displaystyle\int_0^1 x(x^2+1)^3\,dx$ $\quad u = x^2 + 1$

(b) $\displaystyle\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx$ $\quad u = x^2 + 9$

(c) $\displaystyle\int_0^{\sqrt{2}} x\sqrt{x^2+2}\,dx$ $\quad u = x^2 + 2$

(d) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+1}}\,dx$ $\quad u = x^4 + 1$

(e) $\displaystyle\int_0^1 x(x-1)^4\,dx$ $\quad u = x - 1$

(f) $\displaystyle\int_0^1 x(x+1)^4\,dx$ $\quad u = x + 1$

(g) $\displaystyle\int_0^1 x(1-x)^5\,dx$ $\quad u = 1 - x$

(h) $\displaystyle\int_0^1 x^3(x^2+1)^3\,dx$ $\quad u = x^2 + 1$

(i) $\displaystyle\int_0^3 \frac{x}{\sqrt{1+x}}\,dx$ $\quad u = 1 + x$

(j) $\displaystyle\int_0^3 x\sqrt{1+x}\,dx$ $\quad u = 1 + x$

(k) $\displaystyle\int_0^1 \frac{x^3}{1+x^4}\,dx$ $\quad u = 1 + x^4$

(l) $\displaystyle\int_0^1 \frac{x}{1+x^4}\,dx$ $\quad u = x^2$

(m) $\displaystyle\int_0^{\pi/4} \sin^3x\cos x\,dx$ $\quad u = \sin x$

(n) $\displaystyle\int_0^{\pi/4} \cos^3x\sin x\,dx$ $\quad u = \cos x$

(o) $\displaystyle\int_0^{\pi/3} \sec^2x\tan^2x\,dx$ $\quad u = \tan x$

(p) $\displaystyle\int_0^{\pi/4} \sec^2x(1+\tan^2x)\,dx$ $\quad u = \tan x$

INTEGRATION BY CHANGE OF VARIABLE

INDEFINITE INTEGRALS

(The integration constant has been omitted)

(a) $\frac{1}{8}(x^2+4)^4$ (b) $\frac{1}{2}\sqrt{x^4+4}$ (c) $\frac{(2-x)^5}{5}-\frac{(2-x)^4}{2}$

(d) $\frac{2(x+2)^{5/2}}{5}-\frac{4(x+2)^{3/2}}{3}$ (e) $\frac{1}{6}\tan^{-1}(\frac{2x+1}{3})$

(f) $ln(ln\, x)$ (g) $\frac{1}{5}\sin^5 x$ (h) $\frac{1}{2}\tan^2 x$ (i) $\frac{1}{2}e^{x^2}$

(j) $2\tan^{-1}\sqrt{x}$ (k) $\frac{1}{3}\sin^{-1}(\frac{3x-1}{2})$

(l) $\frac{1}{8(2x+1)^2}-\frac{1}{4(2x+1)}$

DEFINITE INTEGRALS

(a) $\frac{15}{8}$ (b) 2 (c) $\frac{8-2\sqrt{2}}{3}$ (d) $\frac{\sqrt{2}-1}{2}$ (e) $\frac{1}{30}$ (f) $\frac{43}{10}$

(g) $\frac{1}{42}$ (h) $\frac{49}{40}$ (i) $\frac{8}{3}$ (j) $\frac{116}{15}$ (k) $\frac{1}{4}ln2$ (l) $\frac{\pi}{8}$ (m) $\frac{1}{16}$

(n) $\frac{3}{16}$ (o) $\sqrt{3}$ (p) $\frac{4}{3}$ (q) $\frac{2}{3}$ (r) $\frac{2}{3}$ (s) $ln\ 2$ (t) $\frac{\pi}{4}$

(u) $e-1$ (v) $\frac{1-e^{-4}}{2}$ (w) $ln\ 2$ (x) $\frac{\pi}{6}$

(y) $ln2-\frac{1}{2}$ (z) $\frac{1-ln2}{2}$

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Integrat^n By Substitut^n 3 Unit Wksht :-

(a) $\int x(x^2+4)^3\,dx$, $u = x^2+4$

$\frac{du}{dx} = 2x$

$= \int x(u^3)\frac{du}{2x}$ $\quad dx = \frac{du}{2x}$

$= \frac{1}{2}\int u^3\,du$

$= \frac{u^4}{8} + c$

$= \frac{(x^2+4)^8}{8} + c$

(b) $\int \frac{x^3}{\sqrt{x^4+4}}\,dx$, $u = x^4+4$

$\frac{du}{dx} = 4x^3$

$dx = \frac{du}{4x^3}$

$= \int \frac{x^3}{\sqrt{u}}\frac{du}{4x^3}$

$= \frac{1}{4}\int u^{-1/2}\,du$

$= \frac{2u^{1/2}}{4} + c$

$= \frac{u^{1/2}}{2} + c$

$= \frac{\sqrt{x^4+4}}{2} + c$

(c) $\int x(2-x)^3\,dx$, $u = 2-x$, $x = 2-u$

$\frac{du}{dx} = -1$

$dx = -du$

$= \int (2-u)(u^3)\,-du$

$= \int -2u^3 + u^4\,du$

$= \frac{-2u^4}{4} + \frac{u^5}{5} + c$

$= \frac{-u^4}{2} + \frac{u^5}{5} + c$

$= \frac{(2-x)^5}{5} - \frac{(2-x)^4}{2} + c$

(d) $\int x\sqrt{x+2}\,dx$, $u = x+2$ $\quad \frac{du}{dx} = 1$

$x = u-2$ $\quad du = dx$

$\int (u-2)(\sqrt{u})\,du$

$= \int u^{3/2} - 2u^{1/2}\,du$

$= \frac{2u^{5/2}}{5} - \frac{4u^{3/2}}{3} + c$

$= \frac{2(x+2)^{5/2}}{5} - \frac{4(x+2)^{3/2}}{3} + c$

(e) $\int \frac{1}{9+(2x+1)^2}\,dx$, $u = 2x+1$

$du/dx = 2$

$dx = \frac{du}{2}$

$= \int \frac{1}{9+u^2}\frac{du}{2}$

$= \frac{1}{2}\left[\frac{1}{3}\tan^{-1}\frac{u}{3}\right]$

$= \frac{1}{6}\tan^{-1}\frac{2x+1}{3} + c$

$= \frac{1}{2}\int \frac{1}{9} + u^{-2}\,du$

$= \frac{1}{2}\left(\frac{x}{9} + \frac{u^{-1}}{-1}\right) + c$

$= \frac{x}{18} - \frac{u^{-1}}{2} + c$

$= \frac{(2x+1)}{18} - \frac{1}{2(2x+1)} + c$

(f) $\int \frac{1}{x\ln x}\,dx \qquad u = \ln x, \quad \frac{du}{dx} = \frac{1}{x}, \quad du = \frac{1}{x}\,dx$

$= \int \frac{1}{\ln x}\cdot\frac{1}{x}\,dx$

$= \int \frac{1}{u}\,du$

$= \ln u + c$

$= \underline{\ln(\ln x) + C}$

(g) $\int \sin^4 x \cos x\,dx. \qquad u = \sin x, \quad \frac{du}{dx} = \cos x, \quad dx = \frac{du}{\cos x}$

$= \int u^4 \cdot \cos x \cdot \frac{du}{\cos x}$

$= \int u^4\,du$ ✓

$= \frac{u^5}{5} + c$

$= \frac{\sin^{\cancel{5}} x}{5} + c$ ✓

(h) $\int \sec^2 x \tan x\,dx, \qquad u = \tan x, \quad \frac{du}{dx} = \sec^2 x$

$= \int \frac{du}{dx}\cdot u \cdot dx$

$= \int u\,du$ ✓

$= \frac{\cancel{k} u^{\cancel{4}2}}{2} + c$

~~[illegible]~~

$= \frac{\tan^2 x}{2} + c$ ✓

(i) $\int x e^{x^2}\,dx, \qquad u = x^2, \quad x = \sqrt{u}. \quad \frac{du}{dx} = 2x$

$= \int x \cdot e^u \cdot \frac{du}{2x}$

$= \frac{1}{2}\int e^u\,du.$ ✓

$= \frac{1}{2} e^{u} + c$

$= \frac{e^{x^2}}{2} + c$ ✓

(j) $\int \frac{1}{\sqrt{x}\,(1+x)}\,dx. \qquad u = \sqrt{x}, \quad x = u^2, \quad \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$= \int \frac{1}{\sqrt{x} + x\sqrt{x}}\,dx.$

$= \int \frac{1}{u + u^2\cdot u}\,\frac{du}{dx}\,\cancel{dx}$

$= \int \frac{1}{u + u^3}\,du.$

$= \int u^{-1} + u^{-3}\,du$

$= u^0$

$= \int \frac{1}{u(1+u^2)}\cdot 2u\,du$

$= 2\int \frac{1}{1+u^2}\,du$

$= 2\tan^{-1} u + c$

$= 2\tan^{-1}\sqrt{x} + c$

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(k) $\displaystyle\int \frac{1}{\sqrt{4-(3x-1)^2}}\,dx \qquad u = 3x-1, \quad \frac{du}{dx} = 3, \quad dx = \frac{du}{3}$

$\displaystyle = \int \frac{1}{\sqrt{4}} - \frac{1}{u^2} \cdot \frac{du}{3}$

$=$

$\displaystyle = \frac{1}{3}\int \frac{1}{2} - \frac{1}{u^2}\,du$

$\displaystyle = \frac{1}{3}\left[\frac{u^2}{2} + u^{-1} + c\right]$

$\displaystyle = \frac{u^2}{6} + \frac{1}{3u} + c$

$\displaystyle = \frac{(3x-1)^2}{6} + \frac{1}{3(3x-1)} + c$

$\displaystyle \int \frac{1}{\sqrt{4-u^2}} \cdot \frac{du}{3}$

$\displaystyle = \frac{1}{3}\int \frac{1}{\sqrt{4-u^2}}\,du$

$\displaystyle = \frac{1}{3}\sin^{-1}\left(\frac{u}{2}\right) + c$

$\displaystyle = \frac{1}{3}\sin^{-1}\left(\frac{3x-1}{2}\right) + c$

(l) $\displaystyle\int \frac{x}{(2x+1)^3}\,dx \qquad u = 2x+1, \quad x = \frac{u-1}{2}, \quad du/dx = 2, \quad dx = \frac{du}{2}$

$\displaystyle = \int \frac{u-1}{2} \times \frac{1}{u^3} \cdot \frac{du}{2}$

$\displaystyle = \int \frac{u-1}{4u^3}\,du$

$\displaystyle = \frac{1}{4}\int \frac{u-1}{u^3}\,du$

$\displaystyle = \frac{1}{4}\int \frac{1}{u^2} - \frac{1}{u^3}\,du$

$\displaystyle = \frac{1}{4}\int u^{-2} - u^{-3}\,du$

$\displaystyle = \frac{1}{4}\left[-u^{-1} + \frac{u^{-2}}{2} + c\right]$

$\displaystyle = \frac{(2x+1)^{-2}}{8} - \frac{(2x+1)^{-1}}{4} + c$

$\displaystyle = \frac{1}{8(2x+1)^2} - \frac{1}{4(2x+1)} + C$

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Definite Integrals:-

(a) $\int_0^1 x(x^2+1)^3 dx$ $u=x^2+1$, $\frac{du}{dx}=2x$, $dx=\frac{du}{2x}$

$= \int_1^2 x \cdot u^3 \cdot \frac{du}{2x}$

$= \frac{1}{2}\int_1^2 u^3 \, du$ ✓

$= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2$

$= \frac{1}{2}\left[4 - \frac{1}{4}\right]$

$= \frac{15}{8}$ // ✓

(b) $\int_0^4 \frac{x}{\sqrt{x^2+9}} dx$ $u=x^2+9$, $\frac{du}{dx}=2x$, $dx=\frac{du}{2x}$

$= \int_9^{25} \frac{x}{\sqrt{u}} \cdot \frac{du}{2x}$

$= \frac{1}{2}\int_9^{25} u^{-1/2} du$ ✓

$= \frac{1}{2}\left[2u^{1/2}\right]_9^{25}$

$= \frac{1}{2}(10-6)$ ✓

$= 2$ //

(c) $\int_0^{\sqrt{2}} x\sqrt{x^2+2}\, dx$ $u=x^2+2$, $\frac{du}{dx}=2x$

$= \int_2^4 x\sqrt{u} \cdot \frac{du}{2x}$

$= \frac{1}{2}\int_2^4 u^{1/2} du$ ✓

$= \frac{1}{2}\left[\frac{2u^{3/2}}{3}\right]_2^4$

$= \frac{1}{2}\left(\frac{16}{3} - \frac{4\sqrt{2}}{3}\right)$

$= \frac{8-2\sqrt{2}}{3}$ ✓

(d) $\int_0^1 \frac{x^3}{\sqrt{x^4+1}} dx$ $u=x^4+1$, $\frac{du}{dx}=4x^3$, $dx=\frac{du}{4x^3}$

$= \int_1^2 \frac{\cancel{x^3}}{\sqrt{u}} \cdot \frac{du}{4\cancel{x^3}}$

$= \frac{1}{4}\int_1^2 u^{-1/2} du$ ✓

$= \frac{1}{4}\left[2u^{1/2}\right]_1^2$

$= \frac{1}{4}(2\sqrt{2}-2)$

$= \frac{\sqrt{2}-1}{2}$ ✓

(e) $\int_0^1 x(x-1)^4 dx$ $x=u+1$, $u=x-1$, $\frac{du}{dx}=1$

$= \int_{-1}^0 (u+1)(u^4)\, du$

$= \int_{-1}^0 u^5 + u^4 \, du$ ✓

$= \frac{u^6}{6} + \frac{u^5}{5}\Big|_{-1}^0$

$= 0 - \left(\frac{1}{6} - \frac{1}{5}\right)$

$= \frac{1}{30}$ // ✓

(f) $\int_0^1 x(x+1)^4 dx$ $x=u-1$, $u=x+1$, $\frac{du}{dx}=1$

$= \int_1^2 (u-1)u^4 \cdot du$

$= \int_1^2 u^5 - u^4 \, du$ ✓

$= \left[\frac{u^6}{6} - \frac{u^5}{5}\right]_1^2$

$= \left(\frac{32}{3} - \frac{32}{5}\right) - \left(\frac{1}{6} - \frac{1}{5}\right)$

$= \frac{43}{10}$ // ✓

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Definite Integrals:-

(g) $\int_0^1 x(1-x)^5\,dx$ $\quad u = 1-x \quad x = 1-u \quad \frac{du}{dx} = -1$

$= \int_1^0 (1-u)u^5 \cdot -du$

$= \int_1^0 -u^5 + u^6\,du$

$= -\frac{u^6}{6} + \frac{u^7}{7}\Big|_1^0$

$= \left(0 - \left(\frac{-1}{6} + \frac{1}{7}\right)\right)$

$= \frac{1}{42}$ //

(h) $\int_0^1 x^3(x^2+1)^3\,dx$, $\quad u = x^2+1 \quad \frac{du}{dx} = 2x \quad x^2 = u-1 \quad dx = \frac{du}{2x}$

$= \int_1^2 x^{\cancel{3}\,2}(u^3)\cdot\frac{du}{2\cancel{x}}$

$= \int_1^2 (u-1)u^3 \cdot \frac{du}{2}$

$= \frac{1}{2}\int_1^2 u^4 - u^3\,du$

$= \frac{1}{2}\left[\frac{u^5}{5} - \frac{u^4}{4}\right]_1^2$

$= \frac{1}{2}\left[\left(\frac{32}{5} - 4\right) - \left(\frac{1}{5} - \frac{1}{4}\right)\right]$

$= \frac{49}{40}$ //

(i) $\int_0^3 \frac{x}{\sqrt{1+x}}\,dx$, $\quad u = 1+x \quad x = u-1 \quad \frac{du}{dx} = 1$

$= \int_1^4 \frac{u-1}{\sqrt{u}}\,du$

$= \int_1^4 u^{1/2} - u^{-1/2}\,du$

$= \left[\frac{2u^{3/2}}{3} - 2u^{1/2}\right]_1^4$

$= \left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - 2\right)$

$= \frac{8}{3}$ //

(k) $\int_0^1 \frac{x^3}{1+x^4}\,dx$ $\quad u = 1+x^4 \quad \frac{du}{dx} = 4x^3$

$= \int_1^2 \frac{\cancel{x^3}}{u}\cdot\frac{du}{4\cancel{x^3}}$

$= \frac{1}{4}\int_1^2 u^{-1}\,du$

$= \frac{1}{4}\left[\ln u\right]_1^2$

$= \frac{1}{4}\ln 2$ //

(l) $\int_0^1 \frac{x}{1+x^4}\,dx$ $\quad u = x^2 \quad \frac{du}{dx} = 2x$

$= \cancel{\int_0^1 x + x^3\,du}$

$= \int_0^1 \frac{\cancel{x}}{1+x^4}\cdot\frac{du}{2\cancel{x}} \quad = \frac{1}{2}\int_0^1 \frac{du}{1+u^2}$

$= \frac{1}{2}\int_0^1 1 + \frac{1}{x^4}\,du \quad = \frac{1}{2}\left[\tan^{-1} u\right]_0^1$

$= \frac{1}{2}\int_0^1 1 + \frac{1}{2u}\,du \quad = \frac{1}{2}\tan^{-1} 1$

$= \frac{1}{2}\left[u + \frac{1}{2}\ln u\right]_0^1 \quad = \frac{1}{2}\times\frac{\pi}{4}$

$= \frac{1}{2}(1 - \quad = \frac{\pi}{8}$.

(m) $\int_0^{\pi/4} \sin^3 x\cos x\,dx$ $\quad u = \sin x \quad \frac{du}{dx} = \cos x$

$= \int_0^{1/\sqrt{2}} u^3\cdot\cancel{\cos x}\cdot\frac{du}{\cancel{\cos x}}$

$= \frac{u^4}{4}\Big|_0^{1/\sqrt{2}}$

$= \frac{1}{16}$ //

(n) $\int_0^{\pi/4} \cos^3 x\sin x\,dx$ $\quad u = \cos x \quad \frac{du}{dx} = -\sin x \quad dx =$

$= \int_1^{1/\sqrt{2}} u^3\cdot\cancel{\sin x}\cdot\frac{du}{-\cancel{\sin x}}$

$= \int_1^{1/\sqrt{2}} -u^3\,du$

$= -\frac{u^4}{4}\Big|_1^{1/\sqrt{2}}$

$= -1 \quad . \quad 1 \qquad 3$

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(o) $\int_0^{\pi/3} \sec^2 x \tan^2 x \, dx$ $\quad u = \tan x$

$\frac{du}{dx} = \sec^2 x$.

$dx = \frac{du}{\sec^2 x}$

$= \int_0^{\sqrt{3}} \frac{du}{\cancel{dx}} \cdot u^2 \cdot \cancel{dx}$ ✓

$= \frac{u^3}{3} \Big|_0^{\sqrt{3}}$ ✓

$= \sqrt{3}$ //

(p) $\int_0^{\pi/4} \sec^2 x \, (1 + \tan^2 x) \, dx$ $\quad u = \tan x$

$du/dx = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

$= \int_0^1 \cancel{\sec^2 x} \, (1 + u^2) \cdot \frac{du}{\cancel{\sec^2 x}}$

$= u + \frac{u^3}{3} \Big|_0^1$ ✓

$= \left(1 + \frac{1}{3}\right)$ ✓

$= \frac{4}{3}$ //