St Catherine's
School
Waverley, Sydney

Student Number:..

Extension I Mathematics
Assessment Task 3
6 June-2007

Time allowed: 55 minutes

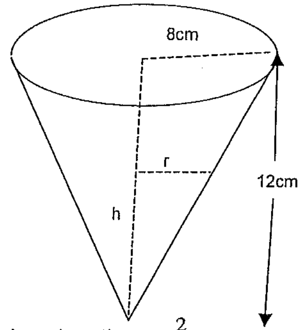
General Instructions

- Attempt ALL questions
- Write your NAME or Student NUMBER at the top of this page and on any extra writing paper used
- Answer the questions in the spaces provided in this paper

Questions	Marks

Question 1.

The diagram shows a parabolic drinking cup of height 12 cm and radius 8 cm. The cup is being filled with water at a constant rate of 20 cm^3 per second. The height of the water at time t seconds is h cm and the radius is r cm.



(i) Using similar triangles, show that $r = \frac{2}{3}h$ (1m)

(ii) Find the rate at which the height is increasing, when the height of the water level is 10 cm. Leave your answer in terms of π.

(Volume is given by $V = \frac{1}{3}\pi r^2 h$) (3m)

Question 2.

The rate at which a body cools in air is assumed to be proportional to the difference between its temperature T and the constant temperature S of the surrounding air. This can be expressed by the differential equation

$$\frac{dT}{dt} = k(T - S),$$

where t is the time in hours and k is a constant.

(i) Show that $T = S + Ae^{kt}$, where A is a constant is a solution of the differential equation. (1m)

A heated body cools from 90^0c to 60^0c in 2 hours. The surrounding air temperature S the body is 25^0C.

(ii) Show that $T = 25 + 65e^{-0.3095t}$ (3m)

(iii) Find the time taken for the body to have a temperature of 30^0C (2m)

(iv) What is the limiting value of the temperature. (1m)

(v) Draw a sketch of the equation relating T in terms of t. (2m)

Question 3.

The velocity of a particle travelling in a straight line is given by $v = 5x$, where x is the displacement from the origin.

(i) Given that initially the particle has a velocity of 5metres per second, find an expression for x, the displacement of the particle in terms of t, the time. (4m)

(ii) Find an expression of acceleration a of the particle in terms of t, the time. (1m)

Question 4.

A ladder 5 metres in length is leaning against a wall. It is slipping down the wall at the rate is 0.2 metres per second. Find the rate at which its foot is slipping on the floor when its foot is 3 metres away from the wall. (4m)

Question 5

The acceleration of a particle in motion is given by

$$\frac{d^2x}{dt^2} = 4 - 2x$$

The particle has a velocity of 4 metres per second at the point x=1.

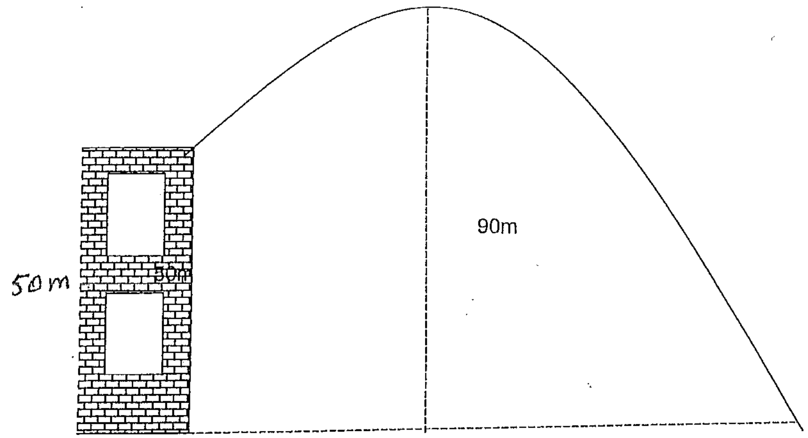
(i) Show that the velocity of the particle is given by $v^2 = 2\,(9 - (x-2)^2)$ (3m)

(ii) Find the centre, the end points and the amplitude of the motion (3m)

(iii) Find the period of motion. (1m)

Question 6.

The diagram shows the path of a projectile launched with a velocity of 40 metres per second at an angle of elevation θ to the horizontal from the top of a building 50 metres high. The acceleration due to gravity is 10 metres per second per second.



(i) Show that

$x = (40\cos\theta)\,t$ and $y = -5t^2 + (40\sin\theta)\,t + 50$ (2m)

(ii) The maximum height reached is 90 metres from the ground. Prove that the angle of projection is 45^0 (3m)

(iii) Find **only** the magnitude of the velocity of the particle when $t = \sqrt{2}$ (2m)

End of Paper.

Question 2

$$\frac{dT}{dt} = k(T-S)$$

(i) $T = S + Ae^{kt}$

$\frac{dT}{dt} = A \cdot k \cdot e^{kt}$ $\frac{1}{2}$

$= kAe^{kt}$

$= k(T-S)$ $\frac{1}{2}$

(ii) $t=0, \; T=90 \qquad t=2, \; T=60$ ($\frac{1}{2}$)

$90 = S + Ae^{0}$

$= 25 + A$

$\therefore A = 65$ (1)

$60 = 25 + 65e^{2k}$ $\frac{1}{2}$

$35 = 65e^{2k}$

$2k = \ln\frac{35}{65}$ $\frac{1}{2}$

$k = \frac{1}{2}\ln\frac{35}{65}$

$= -0.3095$ (2)

$\therefore T = 25 + 65e^{-0.3095t}$ ($\frac{1}{2}$)

(iii) $30 = 25 + 65e^{-0.3095t}$

$\frac{5}{65} = e^{-0.3095t}$ (1)

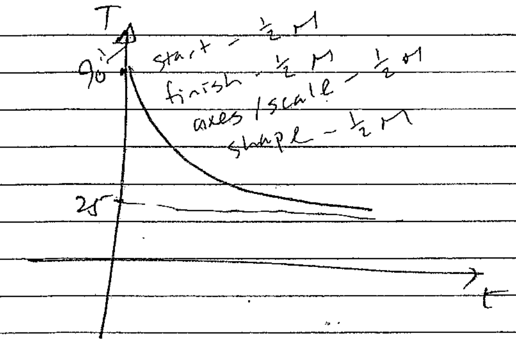
$t = \frac{-1}{0.3095}\ln\frac{5}{65}$

$= 8.29$ hrs. (1)

$t \to \infty; \quad e^{-0.3095t} \to 0$

$\therefore T \to 25$ (1)

(v)



(2)

Q4

$\frac{8}{12} = \frac{r}{h}$

$\therefore r = \frac{8h}{12}$

$= \frac{2}{3}h$

$V = \frac{1}{3}\cdot\pi\cdot\left(\frac{2}{3}h\right)^2\cdot h$

$= \frac{1}{3}\pi\cdot\frac{4}{9}\cdot h^3$ (1)

$= \frac{4\pi}{27}h^3$

$\frac{dV}{dt} = \frac{4\pi}{27}\,3h^2\cdot\frac{dh}{dt}$ (1)

$20 = \frac{4\pi}{27}\times 3\times 10^2\times\frac{dh}{dt}$, ($\frac{1}{2}$)

$$\frac{dr}{dt} = \frac{20 \times 27}{8 \times 10^2 \times 4\pi}$$

$$= \frac{9}{20\pi} \qquad \frac{1}{2}$$

Q.3 $\quad v = 5x$

$$\frac{dt}{dx} = \frac{1}{5x} \qquad (1)$$

$$t = \frac{1}{5}\ln x + c \qquad (1)$$

$t = 0,\ v = 5 \ \therefore x = 1 \qquad \frac{1}{2}$

$$0 = \frac{1}{5}\ln 1 + c$$

$$\therefore c = 0 \qquad \frac{1}{2}$$

$$\therefore t = \frac{1}{5}\ln x \qquad \frac{1}{2}$$

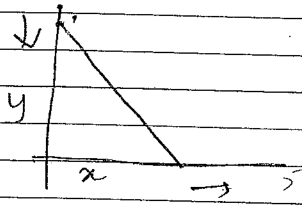
$$\ln x = 5t$$

$$x = e^{5t} \qquad (1)$$

$$\frac{dx}{dt} = 5e^{5t} \qquad \frac{1}{2}$$

$$\frac{d^2x}{dt^2} = 25e^{5t} \qquad \frac{1}{2}$$

Q.4



$$\frac{dy}{dt} = -0.2$$

$$\frac{dx}{dt} = ?$$

$$x = 3$$

$$y = \sqrt{5^2 - 3^2} = 4 \qquad (1)$$

$$5^2 = x^2 + y^2 \qquad (1)$$

$$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \qquad (1)$$

$$0 = 3 \times \frac{dx}{dt} + 4 \times -0.2$$

$$\frac{dx}{dt} = \frac{0.8}{3}\ \text{m/s} \qquad (1)$$

$$= 0.26 = \frac{4}{15}$$

minus sign – deduct $\frac{1}{2}$

Q.5

$$\frac{d^2x}{dt^2} = 4 - 2x$$

$$= -2(x - 2)$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4 - 2x$$

$$\frac{1}{2}v^2 = 4x - x^2 + c \qquad (1)$$

$$\frac{1}{2} \times 4^2 = 4 - 1^2 + c$$

$$c = 5 \qquad (1)$$

$\therefore \frac{1}{2}v^2 = 4x - x^2 + 5$

$v^2 = 8x - 2x^2 + 10$

$= 2(5 - x^2 + 4x)$

$= 2(9 - (x^2 - 4x + 4))$ (1)

$= 2(9 - (x-2)^2)$

(ii). Centre is at $x = 2$, for $\frac{d^2x}{dt^2} = -2(x-2)$ (1/2)

at end pts, $v = 0$ (1)

$9 - (x-2)^2 = 0$

$x - 2 = \pm 3$

$x = 5, -1$ (1)

amplitude is 3 (1/2)

(iv) Period $= \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$ secs. (1)

~~$T = \frac{2\pi}{n}$ $(\frac{1}{2}\pi)$~~

Question 6

$\ddot{x} = 0$

$\dot{x} = \text{const}$

$t = 0$: $\dot{x} = 40\cos\theta$

$\dot{x} = 40\cos\theta$

$x = 40\cos\theta\, t + C$

$x = 0$ when $t = 0$ $\therefore C = 0$

$\therefore x = (40\cos\theta)t$

$\ddot{y} = -10$

$\dot{y} = -10t + C$; $t = 0$, $\dot{y} = 40\sin\theta$

$\dot{y} = -10t + 40\sin\theta$

$y = -5t^2 + 40\sin\theta\, t + C$; $t = 0$, $y = 50$ $\therefore c = 0$

$\therefore y = -5t^2 + 40\sin\theta\, t +$ [illegible]

At max ht, $\dot{y} = 0$. (1/2)

$-10t + 40\sin\theta = 0$

$t = 4\sin\theta$. (1/2)

Sub in y;

$90 = -5(4\sin\theta)^2 + (40\sin\theta)(4\sin\theta) + 50$ (1)

$40 = -80\sin^2\theta + 160\sin^2\theta$

$= 80\sin^2\theta$

$\sin^2\theta = \frac{1}{2}$

$\sin\theta = \pm\frac{1}{\sqrt{2}}$ (1)

$\theta = 45^\circ$. (positive angle or acute angle)

(iii). $\dot{x} = (40\cos\theta)$

$= \frac{40}{\sqrt{2}} = 20\sqrt{2}$ (1/2)

$\dot{y} = -10\sqrt{2} + 40\sin\theta$

$= -10\sqrt{2} + \frac{40}{\sqrt{2}}$

$= 10\sqrt{2}$. (1/2)

$v^2 = \dot{x}^2 + \dot{y}^2$; $800 + 200$

$v = \sqrt{(20\sqrt{2})^2 + (10\sqrt{2})^2} = \sqrt{1000}$

$= 10\sqrt{10}$ (1)