Student Number:_

St. Catherine's School
Waverley

12th June 2009
HSC ASSESSMENT TASK 3
15%

Extension 1 Mathematics

Time allowed: *55 minutes*
Total marks: *34*

INSTRUCTIONS

- Marks for each part of a question are indicated
- All questions should be attempted on the separate paper provided
- Each question should start on a new page.
- Questions 1-4 should be in one booklet, Questions 5-6 should be in a second booklet
- All necessary working should be shown
- Start each question on a new page
- Approved scientific calculators and drawing templates may be used badly arranged work.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right) \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 **Marks**

The velocity of a particle is given by $\frac{dx}{dt} = 5x$. Initially the particle is at $x = 1$.

(i) Find the exact value of the displacement of the particle when $t = 1$. **2.5**

(ii) Find the value of the acceleration when $x = 1$. **1.5**

Question 2 ***(Start a new page)***

The rate at which the population N of a certain species is decreasing is given by the equation $\frac{dN}{dt} = -k(N-1000)$, for some constant k (t is measured in years).

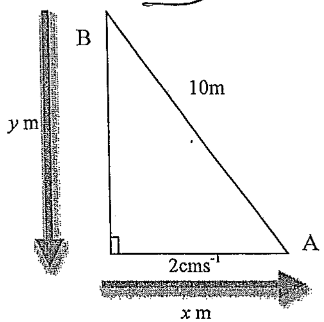
(i) Show that N is given by $N = 1000 + Ae^{-kt}$. 1

(ii) Initially the population is 2500, but after 2 years, there are only 2200 left. Find the values of A and k. (Find k to 3 decimal places) 3

(iii) Find the limiting value of N and hence sketch the graph of population against time. 2

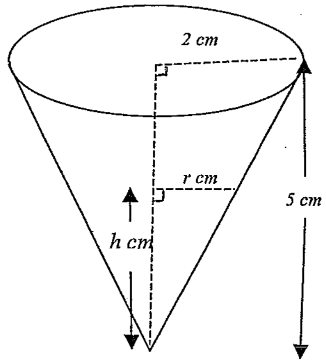
Question 3 ***(Start a new page)***

A ladder represented by AB is 10 metres long. It is placed against a wall with the top B resting on the wall and the foot A is resting on the floor. If the foot is slipping away from the wall at the rate of 2cm per second, find the rate at which the top of the ladder is slipping when the foot is 8 metres from the wall. 4



Question 4 (Start a new page) **Marks**

The ice cream in a cone starts to leak at the rate of **2 cm³** per second. The height of the cone is **5 cm** and the radius is **2 cm.** If at any time t the height is h cm and the radius is r cm.



(i) Show that $h = \frac{5r}{2}$ 1

(ii) Show that the volume, V, of the cone is given by $V = \frac{5\pi\ r^3}{6}$, where r is the radius at any time t. 1

(iii) Find the rate at which the radius is decreasing when $r = 5$ cm. (leave in terms of π) 3

Question 5 **START A NEW BOOKLET** **Marks**

A particle is moving in a straight line. At time t seconds, its velocity v metres per second and displacement x metres is given by the equation:

$$v^2 = 48 + 16x - 4x^2$$

(i) Show that motion is Simple Harmonic and state the centre of motion. **2**

(ii) Find the amplitude of the motion. **3**

(iii) Find the value of the maximum velocity. **1**

(iv) Find the value of the maximum acceleration. **1**

Question 6 ***(Start a new page)***

A golf ball is hit with a velocity of u metres per second at an angle of $30°$ to the horizontal. Taking $g = 10m/\sec^2$:

(i) Show that the equations of the horizontal and vertical components of the motion are given by $x = \dfrac{\sqrt{3}ut}{2}$ and $y = -5t^2 + \dfrac{ut}{2}$, where the axes are placed at the point of projection. **3**

(ii) If the time of flight is 5 seconds, find the value of u. **2**

(iii) Find the range of flight. **1**

(iv) Find the Cartesian equation of motion. **2**

End of Task

Q1) $\frac{dx}{dt} = 5x$

$t=0, x=1$

$\frac{dt}{dx} = \frac{1}{5} \cdot \frac{1}{x}$

$t = \frac{1}{5}\int \frac{1}{x}\, dx$

$= \frac{1}{5} \ln x + c$

$0 = \frac{1}{5} \ln 1 + c$

$c = 0$

$\therefore t = \frac{1}{5} \ln x$

$5t = \ln x$

$x = e^{5t}$

ii) $\dot{x} = 5e^{5t}$

$\ddot{x} = 25e^{5t}$

$\ddot{x} = 25$

Q2)

ii) $2500 = 1000 + Ae^{0}$

$A = 1500$

$2200 = 1000 + 1500e^{-2k}$

$1200 = 1500e^{-2k}$

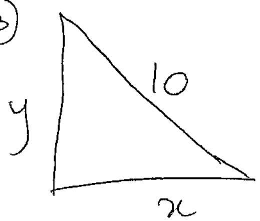
$\frac{1200}{1500} = e^{-2k}$

$-2k = \ln\left(\frac{1200}{1500}\right)$

$k = 0.112$

iii)

Q3



$x^2 + y^2 = 100$

$y = \sqrt{100 - x^2}$

$\frac{dy}{dx} = \frac{1}{\cancel{2}}(100-x^2)^{-\frac{1}{2}} \times -\cancel{2}x$

$= \frac{-x}{\sqrt{100-x^2}}$

$x = 8;\ \frac{dy}{dx} = \frac{-8}{\sqrt{100-64}} = \frac{-8}{6} = -\frac{4}{3}$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$= -\frac{4}{3} \times 0.02 = \frac{-2}{75}$

Q4) ii) $\cancel{\frac{dv}{dt} = \frac{dV}{dt}}$

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$\frac{dV}{dr} = \frac{5}{\cancel{6}2}\pi \cdot \cancel{3}r^2$

$= \frac{5}{2}\pi r^2$

$r=5;\ = \frac{5}{2} \times \pi \times 5^2 = \frac{125\pi}{2}$

$\frac{dr}{dV} = \frac{2}{125\pi}$

$\frac{dr}{dt} = \frac{2}{125\pi} \times \cancel{0.02}\ 2$

$=$

Q5) $v^2 = 48 + 16x - 4x^2$

$\cancel{= 4(12 + 4x}$

$\cancel{= -4(-16 - 4x + x^2)}$

$\cancel{= -4(4 - 4x + x^2)}$

$= 4(12 + 4x - x^2)$

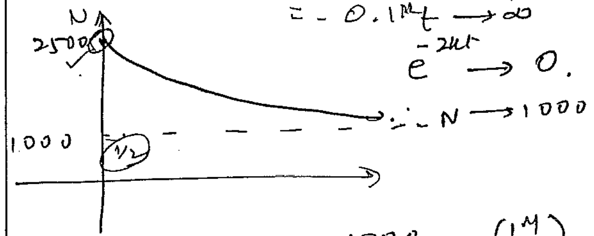
$= 4(16 - 4 + 4x - x^2)$

$= 4(16 - (x-2)^2)$

$= n^2(a^2 - x^2)$

Qn	Solutions	Marks	Comments+Criteria
1.	$\frac{dx}{dt} = 5x$		
	$\frac{dt}{dx} = \frac{1}{5x}$		
	$t = \frac{1}{5}\ln x + c$	1m	(didn't need "c" for the 1m)
	$t=0;\ x=1$		
	$\therefore\ 0 = \frac{1}{5}\ln 1 + c \quad \therefore c = 0$	½ m	
	$t = \frac{1}{5}\ln x \ ;\ \ln x = 5t,\ x = e^{5t}$	½ m	
(i)	$x = e^{5}$ when $t = 1$	½ m	
(ii)	$v = 5x$		
	$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$	½ m	
	$= \frac{d}{dx}\left(\frac{25}{2}x^2\right)$	½ m	
	$= 25x$	½ m.	
	when $x = 1;\ a = 25$		
2	$\frac{dN}{dt} = -k(N - 1000)$		
	$N = 1000 + Ae^{-kt}\ ;\ Ae^{-kt} = N - 1000$		
(i)	$\frac{dN}{dt} = -Ae^{-kt}$	½ m	
	$= -(N - 1000)$	½ m	
(ii)	$t = 0;\ N = 2500$.		
	$2500 = 1000 + Ae^{-k \times 0}$	½ m	
	$\therefore\ A = 1500$	½ m	

Qn	Solutions	Marks	Comments+Criteria
	$t = 2$		
	$N = 2200$		
	$2200 = 1000 + 1500e^{-2k}$ ½		
	$\frac{1200}{1500} = e^{-2k}$ ½	1m	
	$\ln\left(\frac{1200}{1500}\right) = -2k$ ½		
	$\therefore k = -\frac{1}{2}\ln\frac{4}{5}$ ½	1m	Must state k is +ve.
	$= +0.11157m$		-0.5m rounding
	$= -0.1M\ t \to \infty$		
	$e^{-2kt} \to 0.$		



Qn	Solutions	Marks	Comments+Criteria
	$N \to 1000$ (1M)		
	initial 2500 (½m)		
	shape (½m)		
Q.3	$x^2 + y^2 = 10^2$ —— (A)	(½m) (1m)	
	$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$		
	$\frac{dx}{dt} = 2$; $\frac{dy}{dt} = ?$ when $x = 8$	(½m)	-0.5m for using $2ms^{-1}$ or inconsistency in units.
	When $x = 8$ $8^2 + y^2 = 10^2$		
	$y = 6$	(1m)	
	∴ Sub in (A)		
	$8 \times 0.02 + 6 \times \frac{dy}{dt} = 0$		
	$\frac{dy}{dt} = -2/75 \doteq -0.0267 ms^{-1}$		
	or $= -\frac{8}{3}$ cm/s	(1m)	
	$\doteq 2.67 cms^{-1}$		

Qn	Solutions	Marks	Comments+Criteria
	OR		
	$y^2 = 100 - x^2$	(½m)	
	$y = \sqrt{100 - x^2}$		
	$\frac{dy}{dx} = \frac{1}{2\sqrt{100-x^2}} \times -2x$	(½m)	
	$\frac{dy}{dt} = \frac{-x}{\sqrt{100-x^2}} \times \frac{dx}{dt}$	(1m)	
	Sub $x = 8$ and $\frac{dx}{dt} = 2$		
	$\frac{dy}{dt} = \frac{-8}{\sqrt{100-64}} \times 0.02$	(1m)	
	$= -2/75 \doteq 0.0267 ms^{-1}$		
	or $= -\frac{8}{3} cms^{-1}$		
Q.4	using similar triangles		
	$\frac{r}{h} = \frac{2}{5}$ $2h = 5r$		
	$h = \frac{5r}{2}$		
	$\therefore h = \frac{5r}{2}$	1	
	$V = \frac{1}{3} \times \pi r^2 \times h$	½	
	$= \frac{1}{3} \times \pi r^2 \times \frac{5r}{2}$		
	$= \frac{5\pi}{6} r^3$	½	
	$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$	½	
	$= \frac{5\pi}{...} \times 5^3 \times \frac{dr}{dt}$		

Qn	Solutions 1m	Marks	Comments+Criteria
	$\frac{dv}{dt} = \frac{5\pi}{6} \times 3r^2 \times \frac{dr}{dt}$		$-\frac{1}{2}M$ if not in terms of π.
	$2 = \frac{5\pi}{6} \times 3 \times 25 \times \frac{dr}{dt}$	1/2	
	$\frac{dr}{dt} = \frac{12}{375\pi} = \frac{4}{125\pi}$ cms^{-1}	1	
5	$v^2 = 48 + 16x - 4x^2$		
	$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(24 + 8x - 2x^2\right)$	$\frac{1}{2}$m	
	$a = 8 - 4x$	½m ½m	$a = 8 - 4x$ Hence SHM is wrong.
	$a = -4(x-2)$ accel is proportional to the displacement from the centre and is directed towards the centre. Hence SHM.		needs to be rewritten as $-4(x-2)$ to conclude SHM.
(i)	Centre is $x = 2$.	½m	Some here used $v^2 = n^2(a^2 - (x-c)^2)$ to conclude SHM.
(ii)	at the end pts $v = 0$; $v = 0$; $24 + 8x - 2x^2 = 0$	1m	
	$x^2 - 4x - 12 = 0$		
	$(x-6)(x+2) = 0$		
	∴ The end pts are $x = -2$ and $x = 6$ ∴ amplitude is 4 m. 1m.	m	
(iii)	max velocity is at the centre $x = 2$;		
	$v^2 = 48 + 16 \times 2 - 4 \times 2^2$ max vel. $= \|v\| = 8$ ms^{-1} ±?		

Qn	Solutions	Marks	Comments+Criteria
	The max. accel is at an end point $a = 8 - 4(6)$ $= -16$.		
	$\|a\| = 16$ cm/s^2.		
Q.6	$\ddot{x} = 0$ $\ddot{y} = -10$ The only accel. is due to gravity.		
	$\dot{x} = \text{const}$ $\dot{y} = -10t + c$		
	at $t = 0$; $\dot{x} = u\cos 30°$ at $t = 0$; $\dot{y} = u\sin 30° = \frac{u}{2}$		$(-\frac{1}{2}m)$ if constants are not properly evaluated.
	$\therefore \dot{x} = u\cos 30° = u\frac{\sqrt{3}}{2}$ $\therefore \dot{y} = -10t + \frac{u}{2}$		
	$y = -5t^2 + \frac{ut}{2} + c$		
	$x = \frac{u\sqrt{3}}{2}t + c$ at $t = 0$; $y = 0$ ∴ $c = 0$		
	$t = 0$; $x = 0$ ∴ $c = 0$ $y = -5t^2 + \frac{ut}{2}$		
	$\therefore x = \frac{u\sqrt{3}}{2}t$		
(i)	$t = 5$; $y = 0$.		
	$5t^2 = \frac{ut}{2}$		
	$u = 10t = 10 \times 5$ $= 50$ m/s.		
(ii)	$x = ?$ when $t = 5$.		
	$x = \frac{\sqrt{3}}{2} \times 50 \times 5$		
	$= 125\sqrt{3}$ m		
(iv)	$x = \frac{\sqrt{3}u}{2}t$		
	$t = \frac{2x}{\sqrt{3}u} = \frac{x}{25\sqrt{3}}$	1	

Qn	Solutions	Marks	Comments+Criteria
	$y = -5t^2 + 25t \quad (u=50)$ — ② Sub $t = \frac{x}{25\sqrt{3}}$ in ② . $y = -5\left(\frac{x^2}{625 \times 3}\right) + 25\left(\frac{x}{25\sqrt{3}}\right)$ $y = -\frac{x^2}{375} + \frac{\sqrt{3}x}{3}$		