



St Catherine's School

Year: 12

Subject: Extension I Mathematics

Time Allowed: 55 minutes

Date: June 2006

Exam number: 163 612 75

Directions to candidates:

- All questions are to be attempted.
- All necessary **working** must be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Approved calculators and geometrical instruments are required.
- Hand in your work in **1 bundle**:
- Attach the question paper

Extension I Mathematics

Q.1. (i) Find $\int \frac{x dx}{\sqrt{x-5}}$, use the substitution $x-5=u$ (3m)

(ii) Find $\int_0^3 \sqrt{9-x^2} dx$, using the substitution $x=3 \sin \theta$ (5m)

(iii) $\int \frac{x+1}{(x^2+2x-1)^2} dx$, using the substitution $x^2+2x-1=u$ (3m)

Q.2. A spherical balloon is being deflated so that the radius is decreasing at a constant rate of 8 mm per second. Find the rate of change of volume when the radius is 5 mm.

(Note that the volume of a sphere is given by $V = \frac{4\pi r^3}{3}$) (3m)

Q.3. The acceleration of a particle moving in Simple Harmonic motion is given by $\frac{d^2x}{dt^2} = -16x$. The particle starts at the centre of motion with a velocity of 3m/sec

(i) Using only the expression of acceleration and the initial conditions, show that $v^2 = 9 - 16x^2$ (2m)

(ii) Hence or otherwise, find an expression for x in terms of t , where x is the displacement from the origin at time t seconds. (3m)

(iii) Find the value of the acceleration at the end points of the motion. (2m)

P.T.O.

Q.4 . The cooling rate of a body is proportional to the difference between the temperature of the body and that of the surrounding medium and is represented by the equation $\frac{dT}{dt} = -k(T - M)$, where T is the temperature of the body and M is the surrounding temperature.

The original temperature of a body is 90°C and the temperature of the surrounding is 25°C . It cools to 80°C in 20 minutes.

- (i) Show that $T = 25 + Ae^{-kt}$ is a solution to the given equation. (1m)
- (ii) Show that $A = 65$ and $k = 0.0084$ (2 sig figs) (3m)
- (ii) Find the temperature after 30 minutes. (1m)
- (iii) Find the time taken to cool to 30°C . (1m)
- (iii) Sketch the graph of T and find the limiting value of the temperature. (2m)

Q.5 Sue hits a golf ball with a velocity of 50 metres per second and at an angle of α to the horizontal.

- (i) Place the coordinate axes at the point of projection and show that the parametric expressions for x and y, the horizontal and vertical displacements respectively, in terms of t, are given by $x = 50 \cos \alpha t$ and $y = -5t^2 + 50 \sin \alpha t$
(Take g, the acceleration due to gravity as -10m/sec^2) (2m)

- (ii) Find the value of α , the angle of projection, so that the ball just clears a wall 8 metres in height and 10 metres away.

(4m)

END OF PAPER

①

Q1 (i) $\int \frac{x}{\sqrt{x-5}} dx$

Let $x-5=u$

$dx = du$ ✓

$$\int \frac{x}{\sqrt{x-5}} dx = \int \frac{5+u}{\sqrt{u}} du$$

$$= \int (5+u)(u^{-\frac{1}{2}}) du$$

$$= \int 5u^{-\frac{1}{2}} + u^{\frac{1}{2}} du \quad \checkmark$$

$$= \left[\frac{2 \times 5 u^{\frac{1}{2}}}{1} + \frac{2 u^{\frac{3}{2}}}{3} \right]$$

$$= 10\sqrt{u} + 2\sqrt{u^3}$$

~~$2\sqrt{u^3}$~~

$$= 10\sqrt{x-5} + 2\sqrt{(x-5)^3} \quad \checkmark$$

$$= 2(5\sqrt{x-5} + \sqrt{(x-5)^3})$$

(ii) $\int_0^3 \sqrt{9-x^2} dx$

Let $x = 3\sin\theta \Rightarrow \theta = \sin^{-1} \frac{x}{3}$

$dx = 3\cos\theta d\theta$ ✓

When $x=0$, $\theta=0$

$x=3$, $\theta = \frac{\pi}{2}$

$$\int_0^3 \sqrt{9-x^2} dx = \int_0^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta} \cdot 3\cos\theta d\theta$$

$$= 3 \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta$$

$$= \frac{3}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta \quad \checkmark$$

$$= \frac{3}{2} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{3}{2} \left[\frac{\pi}{2} \right] \quad \checkmark = \frac{3\pi^2}{4}$$

②

(iii) $\int \frac{x+1}{(x^2+2x-1)^2} dx$

Let $u = x^2+2x-1$

~~$du = 2x+2$~~

$\frac{du}{dx} = 2x+2 = 2(x+1)$ ✓

$$\int \frac{x+1}{(x^2+2x-1)^2} dx = \frac{1}{2} \int \frac{du}{u^2}$$

$$= \frac{1}{2} \int u^{-2} du \quad \checkmark$$

$$= \frac{1}{2} \left[\frac{u^{-1}}{-1} \right] + c$$

$$= -\frac{1}{2u} + c$$

$$= -\frac{1}{2(x^2+2x-1)} + c \quad \checkmark$$

$$2(x^2+2x-1)$$

Q2

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = 8 \text{ mm/s}$$

$$\frac{dV}{dr} = 4\pi r^2$$

When $r=5$, $\frac{dV}{dr} = 4\pi \times 25 = 100\pi$ ✓

$$\frac{dV}{dt} = 100\pi \times 8$$

$$= 800\pi \text{ mm}^3/\text{s} \quad \checkmark$$

Q3

(i) $\ddot{x} = -16x$

$$\ddot{x} = \frac{d}{dt} \left(\frac{1}{2} v^2 \right)$$

$$\frac{1}{2} v^2 = \int -16x \, dx$$

$$= -8x^2 + c$$

when $x=0, v=3$

$$\frac{1}{2} \times 9 = 0 + c$$

$$c = \frac{9}{2}$$

$$\therefore \frac{1}{2} v^2 = -8x^2 + \frac{9}{2}$$

$$v^2 = -16x^2 + 9$$

(ii) $v = \frac{dx}{dt} = \sqrt{9-16x^2}$

$$\frac{dt}{dx} = \frac{1}{\sqrt{9-16x^2}}$$

$$t = \int \frac{1}{\sqrt{9-16x^2}} \, dx$$

$$= \frac{1}{4} \int \frac{1}{\sqrt{\frac{9}{16} - x^2}} \, dx = \frac{1}{4} \sin^{-1} \left(\frac{4x}{3} \right) + c$$

$$= \frac{1}{4} \cdot \frac{4}{3} \tan^{-1} \frac{4x}{3} + c \quad \text{As } x=0, t=0, c=0$$

$$= \frac{1}{3} \tan^{-1} \frac{4x}{3} + c \quad \therefore t = \frac{1}{3} \tan^{-1} \left(\frac{4x}{3} \right)$$

$$x = \frac{3}{4} \sin 4t$$

when $t=0, x=0$

$$c=0$$

$$12t = \tan^{-1} \frac{4x}{3}$$

$$\frac{4x}{3} = \tan 12t \Rightarrow x = \frac{3}{4} \tan 12t$$

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(iii) At end pt of motion

$$v=0$$

$$9-16x^2=0$$

$$16 \left(\frac{9}{16} - x^2 \right) = 0$$

$$x^2 = \frac{9}{16}$$

$$x = \pm \frac{3}{4}$$

$$a = |-16x|$$

$$= |-16 \left(\frac{3}{4} \right)|$$

$$= |\pm 12 \text{ ms}^{-2}| = 12 \text{ ms}^{-2}$$

4

Q4 $\frac{dT}{dt} = -k(T-M)$

$$t=0, T=90$$

$$M=25$$

$$t=20, T=80$$

(i) $T = 25 + Ae^{-kt}$

$$\frac{dT}{dt} = -k \cdot Ae^{-kt}$$

$$= -k(T-25)$$

$\therefore T = 25 + Ae^{-kt}$ is a sol'n.

(i) $t=0; 90 = 25 + Ae^0$

$$A = 90 - 25 = 65$$

$t=20; 80 = 25 + 65e^{-20k}$

$$55 = 65e^{-20k}$$

$$\frac{55}{65} = e^{-20k}$$

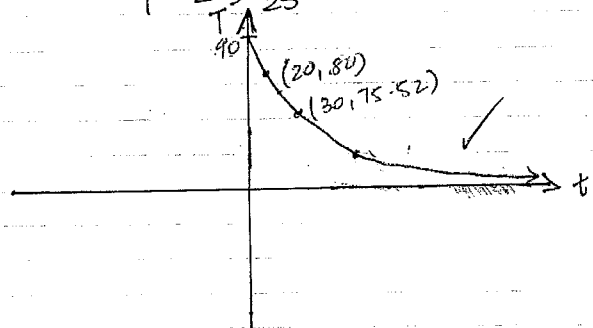
$$k = \frac{-1}{20} \ln \left(\frac{55}{65} \right) = 0.0084 \text{ (asf)}$$

(iii) $t=30, T = 25 + 65e^{-0.0084 \times 30}$
 $= 75.52^\circ$ (2dp)

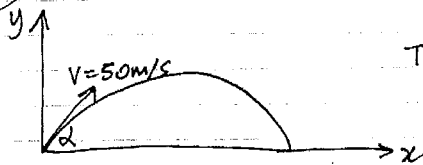
(iv) $T=30, 30 = 25 + 65e^{-0.0084t}$
 $\frac{5}{65} = e^{-0.0084t}$
 $t = \frac{-1}{0.0084} \ln\left(\frac{1}{13}\right)$
 $= 305.35$ mins

(v) as $t \rightarrow \infty$
 $e^{-0.0084t} \rightarrow 0$

$\therefore T \rightarrow 25 + 0$
 $T \rightarrow 25$



(Q5) $V = 50$



The only acceleration is gravity.

$\ddot{x} = 0$
 $\dot{x} = c$
 $t=0, \dot{x} = V \cos \alpha = 50 \cos \alpha$
 $\therefore \dot{x} = 50 \cos \alpha$
 $x = 50t \cos \alpha + c$
 $t=0, x=0 \therefore c=0$
 $\therefore x = 50t \cos \alpha$

$\ddot{y} = -10$
 $\dot{y} = -10t + c$
 $t=0, \dot{y} = 50 \sin \alpha$
 $\therefore c = 50 \sin \alpha$
 $\dot{y} = -10t + 50 \sin \alpha$
 $y = -5t^2 + 50t \sin \alpha + c$
 $t=0, y=0 \therefore c=0$
 $\therefore y = -5t^2 + 50t \sin \alpha$

ii) when $x=10, y=8$

$10 = 50t \cos \alpha \Rightarrow t = \frac{10}{50 \cos \alpha} = \frac{1}{5 \cos \alpha}$

Sub in
 $8 = -5t^2 + 50(\sin \alpha)t$
 $8 = -5\left(\frac{1}{5 \cos \alpha}\right)^2 + 50\left(\frac{1}{5 \cos \alpha}\right) \sin \alpha$
 $= \frac{-1}{5 \cos^2 \alpha} + \frac{10}{\cos \alpha} \cdot \sin \alpha$
 $= -\frac{1}{5} \sec^2 \alpha + 10 \tan \alpha$
 $= -\frac{1}{5} (1 + \tan^2 \alpha) + 10 \tan \alpha$

$-40 = 1 + \tan^2 \alpha - 50 \tan \alpha$
 $\tan^2 \alpha - 50 \tan \alpha + 41 = 0$

$\tan \alpha = \frac{50 \pm \sqrt{50^2 - 4(41)}}{2}$

$\alpha = 88^\circ 50'$ or ~~139.21~~ $39^\circ 50'$