APPLICATION OF CALCULUS TO THE PHYSICAL WORLD

Exercises

1. The mass in grams of a melting iceblock is given by the formula $M = 5 - t^2$ where t is time in minutes. Show that the rate at which the iceblock is melting is given by $R = -2t$ g min^{-1} and find the rate at which it is melting after 2 minutes.
2. The rate of change in velocity over time is given by $\frac{dv}{dt} = 8 + t - t^3$. If the initial velocity is -3 cm s^{-1} find the velocity after 10 s.
3. The rate of flow of wheat into a silo is given by $R = 2 + 6t$ m^{3}h^{-1}. If there is 10 m^3 of wheat initially in the silo, how much is there after 4 hours?
4. The surface area in cm^2 of a balloon being inflated is given by $S = t^3$ where t is time in seconds. Find the rate of increase in the balloon's surface area after 5 s.
5. A point moves along the curve $y = \frac{1}{x}$ such that the x-coordinate changes at a rate of 2 units per second. At what rate is the y-coordinate decreasing when $x = 5$?
6. An ice cube with sides x mm is melting so that its sides are decreasing at 1 mm s^{-1}. What is the rate of decrease in volume when the sides are 150 mm?
7. A cone has a volume given by $V = \frac{3\pi h^2}{10}$, where h is the height of liquid in the cone. If the height of liquid is increasing at a rate of 4 cm s^{-1}, find the rate of increase in volume when the height of liquid is 15 cm.
8. The area of an equilateral triangle is increasing at the rate of 3 cm^2 s^{-1}.
   (a) Show that the area of the triangle is given by $A = \frac{\sqrt{3}x^2}{4}$, where x is the side of the triangle.
   (b) Find the exact rate of increase of the side of the triangle when it has sides of 20 cm.
9. The population in a town is increasing at an annual rate of 5.3%, i.e. $\frac{dP}{dt} = 0.053\,P$.
   If the population is now 36 500, find:
   (a) a formula for the population growth;
   (b) what the population will be in 3 years' time;
   (c) how long it will take for the population to reach 100 000.
10. An object is cooling down according to the formula $T = T_0 e^{-kt}$, where T is temperature in degrees Celsius and t is time in minutes. If the temperature is initially 50°C and the object cools down to 43°C after 15 minutes, find:
    (a) its temperature after an hour;
    (b) how long it takes to cool down to 20°C.
11. The mass (in grams) of a radioactive substance is given by $M = 50\,e^{-0.12t}$ where t is time in years. Find:
    (a) the initial mass;
    (b) the mass after 5 years;
    (c) the rate of decay after 5 years;
    (d) the half-life of the substance (time taken to decay to half its mass).
12. The number of bacteria in a culture increased from 1500 to 3500 in 2 hours. If the rate of bacterial growth is proportional to the number of bacteria at that time, find:
    (a) a formula for the number of bacteria;
    (b) the number of bacteria after 7 hours;
    (c) when there are 1 000 000 bacteria.
13. An object heated to 50°C is placed in a room where the temperature is a constant 22°C.
    (a) If the object cools to 47°C after 2 minutes, show that $T = 22 + 28\,e^{-0.0567t}$.
    (b) When will it reach 25°C?
14. (a) Show that $N = P + Ae^{kt}$ is a solution of
    $$\frac{dN}{dt} = k(N - P),$$
    where k, P and A are constants.
    (b) The number of bacteria in a culture is given by:
    $$\frac{dN}{dt} = k(N - 1000).$$
    If 2500 bacteria increase to 3000 bacteria after 5 hours, find the time taken to double the initial bacteria population.
15. Show that $x = 5 + Ae^{3t}$ is a solution of
    $$\frac{dx}{dt} = 3(x - 5).$$
    If $x = 9$ when $t = 1$, find A to 3 significant figures. Hence find t when $x = 20$.
16. The rate of change of velocity of a particle is given by $\frac{dv}{dt} = -k(v - 10)$, where k is a constant.
    (a) Show that a solution of this differential equation is $v = 10 + Ae^{-kt}$.
    (b) If the initial velocity is 30 m s^{-1} and after 3 s the velocity is 25 m s^{-1}, find the values of A and k.
    (c) Calculate when the velocity reaches 15 m s^{-1}.
    (d) What velocity does the particle tend to as t approaches infinity?

17. If displacement is given by $x = \log_e (t^2 + 3)$, where x is in centimetres and t is in seconds, find:
(a) the acceleration after 2 s;
(b) the initial position of the particle (correct to 3 decimal places);
(c) the exact time when the particle is 5 cm to the right of the origin (O).

18. Displacement of a particle is given by
$$s = t^3 - 4t^2 + 3t,$$
where s is in metres and t is in seconds.
(a) Find the initial velocity.
(b) Find the times when the particle is at the origin.
(c) Find the acceleration after 4 s.

19. The displacement of a particle from the origin is given by $x = e^{3t} + 5$ (x in cm, t in seconds).
(a) Find equations for the velocity $\dot{x}$ and acceleration $\ddot{x}$ of the particle.
(b) Find the initial acceleration.
(c) Show that $\ddot{x} - 5\dot{x} + 6x = 30$.

20. The height of a projectile is given by
$$h = 10 + 3t - t^2,$$
where height is in metres and time is in seconds.
(a) Find the maximum height reached.
(b) Find the velocity after 2 s.
(c) Find the initial height.
(d) How far did the projectile travel in the first 2 s?

21. The acceleration of a particle is given by $a = -10 \sin 5t$ cm s^{-2}. If the initial velocity is 6 cm s^{-1} and the particle is 2 cm to the left of O, find the displacement after π s.

22. The velocity of a particle is given by
$$\dot{x} = \frac{1}{2t + 7} \text{ m s}^{-1}.$$
If the particle is initially at the origin, find its displacement after 3 s (correct to 2 significant figures).

23. The acceleration of an object is given by
$$a = e^t + 12 \text{ m s}^{-2}.$$
If initially $x = 1$ and $v = -2$, find x when $t = 3$ (correct to 3 significant figures).

24. The velocity of a particle is given by
$$v = 4t - 3 \text{ cm s}^{-1}.$$
If the particle is 3 cm from O after 5 s, find the displacement after 10 s.

25. The motion of a particle is given by $\ddot{x} = -e^x$. Given that the velocity is 2 m s^{-1} when x is at the origin, find the velocity when x is 1 m to the right of the origin.

26. The acceleration of a particle which starts at rest 6 cm to the right of O and moves towards O over time t seconds is given by $a = -x^2$.
(a) Find the equation for velocity in terms of the particle's displacement from O.
(b) At what velocity is the particle travelling when it reaches O?

27. The acceleration of a particle is given by
$$\ddot{x} = 2 \cos \pi x \text{ m s}^{-2}.$$
If the particle is initially at the origin with velocity 4 m s^{-1}:
(a) find the velocity when the particle is $\frac{1}{4}$ m from the origin;
(b) show that the particle is never at rest.

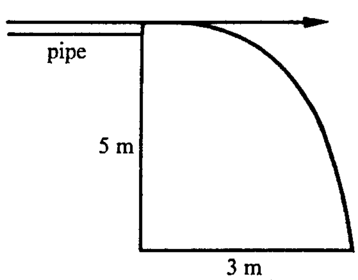
28. The velocity of a particle is given by:
$$\frac{dx}{dt} = 2x \text{ cm s}^{-1}.$$
If the particle is initially at rest 1 cm to the right of the origin:
(a) find the particle's displacement after 2 s;
(b) calculate the velocity after 2 s;
(c) show that the particle is never at the origin;
(d) show that acceleration $a = 4x$.

29. (a) A particle is projected upwards at an angle to the horizontal of α with velocity u. Derive the horizontal and vertical equations for acceleration, velocity and displacement for the flight of the particle.
(b) If $u = 40$ m s^{-1}, $\alpha = 60°$ and $g = 10$ m s^{-2}, find the maximum height reached by the particle (neglecting air resistance).

30. A particle is projected at a velocity of 10 m s^{-1} at an angle of elevation of θ. Neglecting air resistance and using $g = 10$ m s^{-2}, find the Cartesian equation of the displacement of the particle in terms of $\tan \theta$.

31. A ball is thrown from a window which is 100 m from the ground. If the angle of projection is 30° and initial velocity 18 m s^{-1}, find:
(a) the time taken for the ball to land;
(b) how far the ball lands from the base of the building (to the nearest m).

32. A horizontal drainpipe 5 m above sea level empties stormwater into the sea. If the water comes out horizontally and reaches the sea 3 m out from the pipe, find the initial velocity of the water. Let $g = 10$ m s^{-2} and neglect air resistance.



33. A particle's displacement is given by
$x = 5\cos(2t + \frac{\pi}{4})$ cm at time t seconds.
(a) Show that the particle is moving in SHM.
(b) Find the times at which the particle is at the origin.
(c) Write down the period of the motion.
(d) Find the maximum displacement.

34. Show that a particle is moving according to simple harmonic motion if its displacement is given by:
$$x = 3\cos 2t + 4\sin 2t,$$
where x is in metres and t is in seconds.
Find its maximum speed.

35. A particle is moving in SHM and its acceleration is given by
$$\ddot{x} = -x.$$
(a) Show that $x = 7\sin(t + \pi)$ is a possible equation for the displacement of the particle.
(b) Where is the particle when the velocity is $7\ \text{cm s}^{-1}$?
(c) Write down the amplitude and period of the motion.

36. A particle is moving in SHM with acceleration
$$\frac{d^2x}{dt^2} = -4x\ \text{mm s}^{-2}.$$
If the particle starts at the origin with a velocity of $8\ \text{mm s}^{-1}$, find the equation for its velocity in terms of its displacement.

Test yourself

Revision questions

37. The number of organisms in a sample of water from a pond is increasing at a rate proportional to the number of organisms. If there are originally 15 000 organisms, and 3 days later there are 20 000, how many are there after a week?

38. The velocity of a particle is given by
$$v = t^2 - t - 2 \quad \text{m s}^{-1}.$$
Find:
(a) the time when the displacement is minimum;
(b) the acceleration after 3 s.

39. The decay of a substance over t years is given by the formula $\frac{dM}{dt} = -0.3\,M$. If the substance weighs 20 grams, how long will it take to decay to 5 grams?

40. Find an equation for the displacement of a particle from the origin if its acceleration is given by
$$\frac{dv}{dt} = 8t - 6 \quad \text{cm s}^{-2}$$
and initially the particle is at rest 2 cm to the right of the origin.

41. The displacement of a particle is given by
$$x = t^4 - 4t^2 + 4$$
where x is in metres and t is in minutes. Find:
(a) initial displacement and velocity;
(b) acceleration after 5 s;
(c) when the particle is at the origin.

42. The velocity of an object is given by $v = 3\cos 3t$. Show that if the object is initially at the origin, its acceleration is always -9 times its displacement.

43. A rocket is fired from the top of a 50 m cliff. Its initial velocity is $250\ \text{m s}^{-1}$ and its angle of projection is 45°. Neglecting air resistance, show that the equation of its flight is given by:
$$y = -\frac{x^2}{6250} + x + 50 \qquad (g = 10\ \text{m s}^{-2}).$$
Hence find how far the rocket lands from the foot of the cliff.

44. A spherical balloon is being inflated so that the surface area is increasing at a rate of $0.2\ \text{cm}^2\text{s}^{-1}$. Find the rate of increase, (a) in the radius, and (b) in the volume, when the balloon's radius is 2 cm.

45. An electrical charge (Q C) after t s is given by $Q = Q_0e^{-kt}$. Initially the charge is 150 C and after 5 s it is 120 C. Find the charge after 8 s and the rate at which it is discharging at that time.

46. A particle is thrown into the air and travels at a constant acceleration of $-5\ \text{m s}^{-2}$. If the initial velocity is $8\ \text{m s}^{-1}$ and it starts from 2 m above the ground, find its height after 1 s.

Challenge questions

47. The acceleration of a particle is given by

$$\frac{d^2x}{dt^2} = 8 + \frac{4}{t^2},$$

where x is in metres and t is in seconds. If $x = 5$ and $\frac{dx}{dt} = 1$ when $t = 2$, find the displacement after 3 s (in exact form).

48. The population of a town over t years is given by the formula $P = P_0e^{0.0214t}$. How long will it take to double the population?

49. A particle starts at the origin and moves along the x-axis so that its velocity is given by $v = \cos^2 x$ m s^{-1}. When will it reach the point where $x = \frac{\pi}{4}$?

50. The displacement in metres from the origin after t seconds is given by the formula $x = \log_e(\cos t)$. Find:

(a) the initial displacement;
(b) when the particle is at the origin;
(c) the velocity after $\frac{\pi}{4}$ s.

51. I hit a tennis ball with a velocity of 45 m s^{-1} at an angle of projection of α.

(a) Show that the ball just goes over a 4 m fence which is 20 m away when
$20\tan^2\alpha - 405\tan\alpha + 101 = 0$.
(b) In what range of values for α must the ball be hit to go over the fence? (Neglect air resistance and use 10 m s^{-2} for the acceleration due to gravity.)

52. The acceleration of a particle is given by $a = 16\,e^{2x}$. If the particle is initially at the origin and moving at a velocity of 4 cm s^{-1}, find:

(a) its velocity in terms of x;
(b) its displacement as a function of t.

53. A particle is moving with a displacement of $x = e^{3t} + 2$. Show that $\ddot{x} - 2\dot{x} - 3x + 6 = 0$.

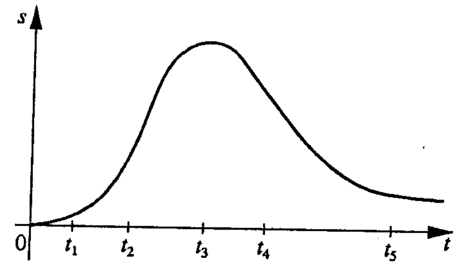
54. A particle has a velocity given by $\dot{x} = \tan t$ m s^{-1}.

(a) Find the acceleration of the particle after $\frac{\pi}{3}$ s.
(b) Use Simpson's Rule with 3 function values to find the approximate value of $\int_0^{\frac{\pi}{3}} \tan t\, dt$ in terms of π. What information does this result give about the particle?

55. A car starts a journey and increases speed at a constant rate until it is travelling at 20 m s^{-1} after 5 s. It then travels at this speed for 10 more seconds until it slows down at a constant rate to 10 m s^{-1} after 5 more seconds.

(a) Graph the speed (v) of the car as a function of time t in seconds.
(b) On a separate diagram, graph the distance (s) travelled by the car as a function of time t.

56. A particle moves in a straight line. The graph shows the distance s of the particle from a fixed point at time t seconds.



(a) What is the velocity at time t_5? Why?
(b) Is the particle moving faster at time t_1 or t_2?
(c) Draw the graph of the velocity of the particle as a function of t.

Rates of change

1. Melting rate 4 g min^{-1}.

2. $\frac{dv}{dt} = 8 + t - t^3,$

$\therefore v = \int (8 + t - t^3)\,dt,$

$= 8t + \frac{t^2}{2} - \frac{t^4}{4} + C.$

When $t = 0, v = -3$

$-3 = 8(0) + \frac{0^2}{2} - \frac{0^4}{2} + C$

$-3 = C,$

$\therefore v = 8t + \frac{t^2}{2} - \frac{t^4}{4} - 3.$

When $t = 10$

$v = 8(10) + \frac{10^2}{2} - \frac{10^4}{4} - 3$

$= -2373 \text{ cm s}^{-1}$

3. 66 m^3

4. $\frac{dS}{dt} = 3t^2$

When $t = 5$

$\frac{dS}{dt} = 3(5^2) = 75.$

Surface area is increasing at a rate of 75 cm^2s^{-1}.

Further rates of change (3 unit)

5. 0.08 units s^{-1}.

6. $V = x^3$

$\frac{dV}{dt} = 3x^2$

$\frac{dx}{dt} = -1$

$\frac{dV}{dt} = \frac{dV}{dx}\cdot\frac{dx}{dt}$

$= 3x^2(-1).$

When $x = 150$

$\frac{dV}{dt} = 3(150^2)(-1)$

$= -67\,500,$

$\therefore$ rate of decrease is 67 500 mm^3s^{-1}.

7. 113 cm^3s^{-1}

8. (a) All sides x and all angles 60°.

$A = \frac{1}{2}ab\sin C$

$= \frac{1}{2}x^2 \sin 60°$

$= \frac{x^2}{2}\cdot\frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}x^2}{4}$

(b) $A = \frac{\sqrt{3}x^2}{4}$

$\frac{dA}{dx} = \frac{2\sqrt{3}x}{4} = \frac{\sqrt{3}x}{2}$

$\frac{dA}{dt} = 3$

$\frac{dA}{dt} = \frac{dA}{dx}\cdot\frac{dx}{dt}$

$3 = \frac{\sqrt{3}x}{2}\cdot\frac{dx}{dt},$

$\therefore \frac{dx}{dt} = \frac{6}{\sqrt{3}x}.$

When $x = 20$

$\frac{dx}{dt} = \frac{6}{\sqrt{3}(20)}$

$= \frac{\sqrt{3}}{10} \text{cm s}^{-1}$

Exponential growth and decay

9. (a) $P = 36\,500\, e^{0.053t}$

(b) 42 790

(c) 19 years

10. When $t = 0, T = 50$, so $T_0 = 50$,

$\therefore T = 50e^{-kt}.$

When $t = 15, T = 43$

$43 = 50e^{-15k}$

$\frac{43}{50} = e^{-15k}$

$0.86 = e^{-15k}$

$\ln 0.86 = \ln e^{-15k}$

$= -15k \ln e$

$= -15k$

$\frac{\ln 0.86}{-15} = k$

$0.01 \doteqdot k.$

So $T = 50\,e^{-0.01t}$

(a) When $t = 60$

$T = 50\,e^{-0.01t}$

$= 50\,e^{-0.01(60)}$

$\doteqdot 27.4.$

So temperature after an hour is 27.4°C.

(b) When $T = 20$

$T = 50\,e^{-0.01t}$

$20 = 50\,e^{-0.01t}$

$\frac{20}{50} = e^{-0.01t}$

$0.4 = e^{-0.01t}$

$\ln 0.4 = \ln e^{-0.01t}$

$= -0.01t \ln e$

$= -0.01t$

$\frac{\ln 0.4}{-0.01} = t$

$91.1 \doteqdot t.$

So it takes 91.1 minutes to cool down to 20°C.

11. (a) 50 g

(b) 27.4 g

(c) $\frac{dM}{dt} = -3.3$ (decaying at the rate of 3.3 g per year).

(d) 5.8 years

12. (a) $N = N_0e^{kt}$

Initially $N = 1500,$

$\therefore N = 1500e^{kt}.$

When $t = 2, N = 3500.$

$3500 = 1500e^{2k}$

$\frac{3500}{1500} = e^{2k}$

$$2.3 = e^{2k}$$
$$\ln 2.3 = \ln e^{2k} = 2k \ln e = 2k$$
$$\frac{\ln 2.3}{2} = k$$
$$0.424 \doteqdot k,$$
$$\therefore N = 1500\, e^{0.424t}.$$

(b) When $t = 7$

$$N = 1500\, e^{0.424(7)} \doteqdot 29\,108.$$

So there are 29 108 bacteria after 7 hours.

(c) When $N = 1\,000\,000$

$$1\,000\,000 = 1500\, e^{0.424t}$$
$$\frac{1\,000\,000}{1500} = e^{0.424t}$$
$$666.7 = e^{0.424t}$$
$$\ln 666.7 = \ln e^{0.424t} = 0.424t \ln e = 0.424t$$
$$\frac{\ln 666.7}{0.424} = t$$
$$15.3 \doteqdot t.$$

So there are 1 000 000 bacteria after 15.3 hours.

Further growth and decay (3 unit)

13. (a) $P = 22$ in the formula $T = P + Ae^{kt}$

(b) 39.4 minutes

14. (a) $N = P + Ae^{kt}$

$$\frac{dN}{dt} = kAe^{kt} = k(P + Ae^{kt} - P) = k(N - P)$$

(b) $N = P + Ae^{kt}$ where $P = 1000$.

When $t = 0$, $N = 2500$

$$2500 = 1000 + Ae^0$$
$$1500 = A,$$
$$\therefore N = 1000 + 1500\, e^{kt}.$$

When $t = 5$, $N = 3000$

$$3000 = 1000 + 1500e^{5k}$$
$$2000 = 1500\, e^{5k}$$
$$1.33 = e^{5k}$$
$$\ln 1.33 = \ln e^{5k}$$
$$0.2877 \doteqdot 5k \ln e = 5k$$
$$0.0575 = k,$$
$$\therefore N = 1000 + 1500\, e^{0.0575t}.$$

For double N, $N = 5000$

$$5000 = 1000 + 1500\, e^{0.0575t}$$
$$4000 = 1500\, e^{0.0575t}$$
$$2.67 = e^{0.0575t}$$
$$\ln 2.67 = \ln e^{0.0575t}$$
$$0.9808 = 0.0575t \ln e = 0.0575t$$
$$17 \doteqdot t.$$

So it takes 17 hours to double the initial number of bacteria.

15. $A \doteqdot 0.199$, $t \doteqdot 1.44$

16. (a) $v = 10 + Ae^{-kt}$

$$\frac{dv}{dt} = -kAe^{-kt} = -k(10 + Ae^{-kt} - 10) = -k(v - 10)$$

(b) When $t = 0$, $v = 30$

$$30 = 10 + Ae^{0}$$
$$20 = A,$$
$$\therefore v = 10 + 20\, e^{-kt}.$$

When $t = 3$, $v = 25$

$$25 = 10 + 20\, e^{-3k}$$
$$15 = 20\, e^{-3k}$$
$$0.75 = e^{-3k}$$
$$\ln 0.75 = \ln e^{-3k}$$
$$-0.288 = -3k \ln e = -3k$$
$$0.0959 \doteqdot k,$$
$$\therefore v = 10 + 20\, e^{-0.0959t}.$$

(c) When $v = 15$:

$$15 = 10 + 20\, e^{-0.0959t}.$$
$$5 = 20\, e^{-0.0959t}$$
$$0.25 = e^{-0.0959t}$$
$$\ln 0.25 = \ln e^{-0.0959t}$$
$$-1.386 = -0.0959t \ln e = -0.0959t$$
$$14.5 \doteqdot t.$$

So velocity reaches 15 m s^{-1} after 14.5 s.

(d) $v = 10 + 20\, e^{-0.0959t}$

As t approaches infinity, $e^{-0.0959t}$ approaches 0, so v approaches 10 m s^{-1}.

Motion and differentiation

17. (a) $a = \frac{-2}{49}$ cm s^{-2}

(b) $x = 1.099$ cm

(c) $t = \sqrt{e^5 - 3}$

18. (a) $v = \frac{ds}{dt} = 3t^2 - 8t + 3$

When $t = 0$, $v = 3(0)^2 - 8(0) + 3 = 3.$

So initial velocity is 3 m s^{-1}.

(b) At the origin, $s = 0$,

i.e. $t^3 - 4t^2 + 3t = 0$

$$t(t^2 - 4t + 3) = 0$$
$$t(t - 3)(t - 1) = 0,$$
$$\therefore t = 0, 1, 3.$$

So the particle is at the origin initially, after 1 second, and after 3 seconds.

(c) $a = \frac{d^2s}{dt^2} = 6t - 8.$

When $t = 4$, $a = 6(4) - 8 = 16.$

So acceleration after 4 seconds is 16 m s^{-2}.

19. (a) $\dot{x} = 3e^{3t}$, $\ddot{x} = 9e^{3t}$

(b) 9 cm s^{-2}

(c) Show that $9e^{3t} - 5(3e^{3t}) + 6(e^{3t} + 5) = 30$

20. (a) For maximum h, $\frac{dh}{dt} = 0$,

i.e. $3 - 2t = 0$

$3 = 2t$

$1.5 = t$

$\left(\frac{d^2h}{dt^2} = -2 < 0, \text{ so } h \text{ is a maximum}\right)$

Maximum height occurs when $t = 1.5$ seconds.

$h = 10 + 3t - t^2$
$= 10 + 3(1.5) - (1.5)^2$
$= 10 + 4.5 - 2.25$
$= 12.25.$

So maximum height is 12.25 m.

(b) $v = \frac{dh}{dt} = 3 - 2t$

When $t = 2$, $v = 3 - 2(2)$
$= -1.$

So velocity is -1 m s^{-1} after 2 seconds.

(c) $h = 10 + 3t - t^2$

Initially $t = 0$,

i.e. $h = 10 + 3(0) - 0^2$
$= 10.$

So initial height is 10 m.

(d) When $t = 2$, $h = 10 + 3(2) - 2^2$
$= 12,$

so initially the particle is 10 m from 0 and after $1\frac{1}{2}$ seconds it is 12.25 m from 0. Therefore it travels 2.25 m in the first $1\frac{1}{2}$ seconds. When $t = 2$, $h = 12$, so it then travels another 0.25 m. Therefore total distance is 2.5 m.

Motion and integration

21. $(4\pi - 2)$ cm

22. $\dot{x} = \frac{1}{2t + 7}$

$x = \int v\, dt$
$= \int \frac{1}{2t + 7}\, dt$
$= \frac{1}{2}\int \frac{2}{2t + 7}\, dt$
$= \frac{1}{2}\log_e(2t + 7) + C$

When $t = 0, x = 0$

$0 = \frac{1}{2}\log_e 7 + C,$

$\therefore C = -\frac{1}{2}\log_e 7,$

$\therefore x = \frac{1}{2}\log_e(2t + 7) - \frac{1}{2}\log_e 7.$

When $t = 3$, $x = \frac{1}{2}\log_e[2(3) + 7] - \frac{1}{2}\log_e 7$
$= \frac{1}{2}\log_e 13 - \frac{1}{2}\log_e 7$
$= \frac{1}{2}\log_e \frac{13}{7}$
$\doteqdot 0.31.$

So displacement after 3 seconds is 0.31 m.

23. 65.1 m

24. $x = \int (4t - 3)\, dt$
$= 2t^2 - 3t + C$

When $t = 5, x = 3,$

so $3 = 2(5)^2 - 3(5) + C$
$= 50 - 15 + C$

$-32 = C,$

$\therefore x = 2t^2 - 3t - 32.$

When $t = 10,$

$x = 2(10)^2 - 3(10) - 32$
$= 138.$

So displacement after 10 s is 138 cm.

Velocity and acceleration in terms of x (3 unit)

25. 0.75 m s^{-1}

26. (a) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -x^2$

$\frac{1}{2}v^2 = \int -x^2\, dx$
$= -\frac{x^3}{3} + C$

When $x = 6$, $v = 0$

$0 = -\frac{6^3}{3} + C$

$72 = C,$

$\therefore \frac{1}{2}v^2 = -\frac{x^3}{3} + 72,$

$v^2 = -\frac{2x^3}{3} + 144,$

$v = \pm\sqrt{-\frac{2x^3}{3} + 144}.$

Since the particle starts 6 cm to the right of O and moves towards O, its velocity is negative,

$\therefore v = -\sqrt{-\frac{2x^3}{3} + 144}.$

(b) When $x = 0$

$v = -\sqrt{-\frac{2(0^3)}{3} + 144}.$
$= -12.$

So velocity is -12 cm s^{-1}.

27. (a) 4.11 m s^{-1}

(b) $\sin(\pi x) = -4\pi$ has no solutions, so $v \neq 0$.

28. $\frac{dx}{dt} = 2x,$

$\therefore \frac{dt}{dx} = \frac{1}{2x}.$

(a) $t = \int \frac{1}{2x}\, dx$
$= \frac{1}{2}\log_e x + C$

When $t = 0, x = 1$

$0 = \frac{1}{2}\log_e 1 + C$

$0 = C,$

$\therefore t = \frac{1}{2}\log_e x,$

$2t = \log_e x,$

$\therefore x = e^{2t}.$

When $t = 2$

$x = e^4$

$\doteqdot 54.6$ cm

(b) $v = 2x$

When $t = 2,\ x = 54.6$

$v = 109.2\ \text{cm s}^{-1}$

(c) $0 = e^{2t}$ has no solution,

$\therefore x \neq 0.$

(d) $\frac{dx}{dt} = 2e^{2t}$

$\frac{d^2x}{dt^2} = 4e^{2t}$

$= 4x,$

$\therefore a = 4x.$

Projectiles (3 unit)

29. (a) $\ddot{x} = 0,\ \dot{x} = u\cos\alpha,\ x = ut\cos\alpha$

$\ddot{y} = -g,\ \dot{y} = -gt + u\sin\alpha,\ y = -\frac{gt^2}{2} + ut\sin\alpha$

(b) Maximum height is 60 m.

30. $\ddot{x} = 0,$

$\therefore \dot{x} = C$

$= 10\cos\theta$

$x = \int (10\cos\theta)\,dt$

$= 10t\cos\theta + k.$

When $t = 0, x = 0$

$0 = 10(0)\cos\theta + k$

$= k,$

$\therefore x = 10t\cos\theta.$ (1)

$\ddot{y} = -10$

$\dot{y} = \int(-10)\,dt$

$= -10t + k.$

When $t = 0, \dot{y} = 10\sin\theta$

$10\sin\theta = -10(0) + k$

$= k,$

$\therefore \dot{y} = -10t + 10\sin\theta,$

$y = \int(-10t + 10\sin\theta)\,dt,$

$= -5t^2 + 10t\sin\theta + k.$

When $t = 0, y = 0$

$0 = -5(0^2) + 10(0)\sin\theta + k$

$= k,$

$\therefore y = -5t^2 + 10t\sin\theta.$ (2)

From (1), $t = \frac{x}{10\cos\theta}.$

Substitute into (2):

$y = -5\left(\frac{x}{10\cos\theta}\right)^2 + 10\left(\frac{x}{10\cos\theta}\right)\sin\theta$

$= \frac{-5x^2}{100\cos^2\theta} + \frac{x\sin\theta}{\cos\theta}$

$= -\frac{x^2}{20}\sec^2\theta + x\tan\theta$

$= -\frac{x^2}{20}(\tan^2\theta + 1) + x\tan\theta$

31. (a) 5.5 s

(b) 85 m

32. Angle of projection is 0°,
so initially:

$\dot{x} = V\cos 0°$

$= V,$

$\therefore x = Vt.$ (1)

Also initially, $\dot{y} = V\sin 0°$

$= 0.$

$\ddot{y} = -10$

$\dot{y} = -10t + k.$

When $t = 0,\ \dot{y} = 0$

$0 = -10(0) + k$

$= k,$

$\therefore \dot{y} = -10t$

$y = -5t^2 + k.$

When $t = 0,\ y = 5$

$5 = -5(0^2) + k$

$= k,$

$\therefore y = -5t^2 + 5.$ (2)

When $x = 3,\ y = 0.$

From (1): $t = \frac{x}{V}$

$= \frac{3}{V}.$

(Note that V must be positive since t is positive.)

When $t = \frac{3}{V}, y = 0.$

From (2): $0 = -5\left(\frac{3}{V}\right)^2 + 5$

$= -\frac{45}{V^2} + 5$

$= -45 + 5V^2$

$45 = 5V^2$

$9 = V^2$

$3 = V$ (V is positive)

So the initial velocity is $3\ \text{m s}^{-1}$.

Simple harmonic motion (3 unit)

33. (a) $\frac{d^2x}{dt^2} = -4x$

(b) $t = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \ldots$ seconds.

(c) Period is π.

(d) Maximum displacement is 5 cm.

34. $x = 3\cos 2t + 4\sin 2t$

$\dot{x} = -6\sin 2t + 8\cos 2t$

$\ddot{x} = -12\cos 2t - 16\sin 2t$

$= -4(3\cos 2t + 4\sin 2t)$

$= -4x,$

$\therefore$ the particle is moving in SHM.

For maximum speed, $\ddot{x} = 0$

$-12\cos 2t - 16\sin 2t = 0$

$\tan 2t = -0.75$

$2t = -0.64$

$\dot{x} = -6\sin(-0.64) + 8\cos(-0.64)$

$= 10$

So maximum speed is 10 m s^{-1}

35. (a) Differentiate $x = 7\sin(x + \pi)$.

(b) At the origin, $x = 0$.

(c) Amplitude is 7.
Period is 2π.

36. $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x$

$\frac{1}{2}v^2 = \int -4x\,dx$

$= -2x^2 + k$

When $x = 0$, $v = 8$

$\frac{1}{2}(8^2) = -2(0^2) + k$

$32 = k,$

$\therefore \frac{1}{2}v^2 = -2x^2 + 32,$

$v^2 = -4x^2 + 64,$

$v = \pm\sqrt{-4x^2 + 64}.$

Condition $x = 0$, $v = 8$ is satisfied by

$v = \sqrt{-4x^2 + 64}.$

Revision questions

37. 29 350 organisms

38. (a) For minimum displacement, $v = 0$,

i.e. $t^2 - t - 2 = 0$

$(t - 2)(t + 1) = 0$

$t = 2$ or -1

$\frac{dv}{dt} = 2t - 1.$

When $t = 2$, $\frac{dv}{dt} = 2(2) - 1 > 0$ (minimum).

So minimum displacement occurs after 2 seconds.

(b) $a = 2t - 1$

When $t = 3$, $a = 2(3) - 1$

$= 5.$

So acceleration after 3 seconds is 5 m s^{-2}.

39. 4.62 years ($M_0 = 20$ and $k = 0.3$)

40. $v = \int (8t - 6)\,dt$

$= 4t^2 - 6t + C$

When $t = 0$, $v = 0$

$0 = 4(0)^2 - 6(0) + C$

$= C,$

$\therefore v = 4t^2 - 6t.$

$x = \int (4t^2 - 6t)\,dt$

$= \frac{4t^3}{3} - 3t^2 + k.$

When $t = 0$, $x = 2$

$2 = \frac{4(0)^3}{3} - 3(0)^2 + k$

$= k,$

$\therefore x = \frac{4t^3}{3} - 3t^2 + 2.$

41. (a) Initial displacement = 4 m
Initial velocity = 0 m min^{-1}

(b) $a = 292$ m min^{-2}

(c) $t = 1.41$ min

42. $x = \int 3\cos 3t\,dt$

$= \sin 3t + C$

When $t = 0$, $x = 0$

$0 = \sin 0 + C$

$= C,$

$\therefore x = \sin 3t,$

$a = \frac{dv}{dt} = -9\sin 3t,$

$= -9x$ (since $x = \sin 3t$).

43. Rocket lands 6299.6 m (6.2996 km) from the foot of the cliff.

44. (a) $S = 4\pi r^2$

$\frac{dS}{dr} = 8\pi r$

$\frac{dS}{dt} = 0.2\ \text{cm}^2\,\text{s}^{-1}$

$\frac{dS}{dt} = \frac{dS}{dr}\cdot\frac{dr}{dt}$

$0.2 = 8\pi r\frac{dr}{dt},$

$\therefore \frac{dr}{dt} = \frac{0.2}{8\pi r}.$

When $r = 2$

$\frac{dr}{dt} = \frac{0.2}{8\pi(2)}$

$\doteqdot 0.004.$

So rate of increase is 0.004 cm s^{-1}.

(b) $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

When $r = 2, \frac{dr}{dt} = 0.004$

$$\frac{dV}{dt} = \frac{dV}{dr}.\frac{dr}{dt}$$
$$= 4\pi r^2 (0.004)$$
$$= 4\pi(2^2)(0.004)$$
$$\doteqdot 0.2.$$

So rate of change of volume is $0.2 \text{ cm}^3\text{s}^{-1}$.

45. Charge is 105 coulombs after 8 s.
Rate of discharge is 4.68 C/s.

46. $v = \int -5\, dt$
$= -5t + k$

When $t = 0,\ v = 8$
$8 = -5(0) + k$
$= k,$
$\therefore v = -5t + 8,$

$h = \int (-5t + 8)\, dt,$
$= -\frac{5t^2}{2} + 8t + C.$

When $t = 0,\ h = 2$
$2 = -\frac{5(0^2)}{2} + 8(0) + C$
$= C,$
$\therefore h = -\frac{5t^2}{2} + 8t + 2.$

When $t = 1$
$h = -\frac{5(1^2)}{2} + 8(1) + 2$
$= 7\frac{1}{2}.$

So the height is $7\frac{1}{2}$ m after 1 s.

Challenge questions

47. $x = 4t^2 - 4\log_e t - 13t + 15 + 4\log_e 2.$
After 3 s, displacement is
$12 + 4(\log_e 2 - \log_e 3)$ m.

48. To double P, $P = 2P_0$.
$$P = P_0 e^{0.0214t}$$
$$2P_0 = P_0 e^{0.0214t}$$
$$2 = e^{0.0214t}$$
$$\ln 2 = \ln e^{0.0214t}$$
$$= 0.0214t \ln e$$
$$= 0.0214t$$
$$\frac{\ln 2}{0.0214} = t$$
$$32.4 \doteqdot t.$$

So the population will double in 32.4 years.

49. 1 s

50. (a) When $t = 0,\ x = \log_e(\cos 0)$
$= \log_e 1$
$= 0.$

So initially the particle is at the origin.

(b) At the origin $x = 0$,
i.e. $0 = \log_e(\cos t)$,
$\therefore \cos t = e^0$
$= 1$
$t = 0, 2\pi, 4\pi, 6\pi, \ldots$

(c) $v = \frac{dx}{dt} = \frac{-\sin t}{\cos t}$
$= -\tan t$

When $t = \frac{\pi}{4},\ v = -\tan\frac{\pi}{4}$
$= -1.$

So the velocity is -1 m s^{-1} after $\frac{\pi}{4}$ s.

51. (a) Find the Cartesian equation for x and y; substitute $x = 20, y = 4$.
(b) $14°10' \le \alpha \le 87°08'$

52. (a) $\frac{d}{dx}(\frac{1}{2}v^2) = 16e^{2x}$
$\frac{1}{2}v^2 = \int 16e^{2x}\, dx$
$= 8e^{2x} + C$

When $x = 0,\ v = 4$
$\frac{1}{2}(4^2) = 8e^0 + C$
$8 = 8 + C$
$0 = C,$
$\therefore \frac{1}{2}v^2 = 8e^{2x}$
$v^2 = 16e^{2x}$
$v = \pm\sqrt{16e^{2x}}$
$= \pm 4e^x.$

The condition $x = 0$ when $v = 4$ is satisfied by $v = 4e^x$.

(b) $\frac{dx}{dt} = 4e^x,$
$\therefore \frac{dt}{dx} = \frac{1}{4e^x}$
$= \frac{1}{4}e^{-x}$
$t = \int \frac{1}{4}e^{-x}\, dx$
$= -\frac{1}{4}e^{-x} + C.$

When $t = 0,\ x = 0$
$0 = -\frac{1}{4}e^0 + C$
$C = \frac{1}{4},$
$\therefore t = -\frac{1}{4}e^{-x} + \frac{1}{4}$
$4t = -e^{-x} + 1$
$e^{-x} = 1 - 4t$
$\frac{1}{e^x} = 1 - 4t$
$e^x = \frac{1}{1 - 4t},$
$\therefore x = \log_e \frac{1}{1 - 4t}.$

53. Show $\ddot{x} - 2\dot{x} - 3x + 6$
$= 9e^{3t} - 2(3e^{3t}) - 3(e^{3t} + 2) + 6$
$= 0$

54. (a) $\ddot{x} = \sec^2 t$

When $t = \frac{\pi}{3},$
$\ddot{x} = \sec^2\frac{\pi}{3}$
$= 2^2$
$= 4 \text{ m s}^{-2}$

(b) $\int_a^b f(x)\,dx$

$$\doteqdot \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$\int_0^{\frac{\pi}{3}} \tan t\,dt$$

$$\doteqdot \frac{\frac{\pi}{3}-0}{6}\left[f(0) + 4f\left(\frac{\pi}{6}\right) + f\left(\frac{\pi}{3}\right)\right]$$

$$= \frac{\pi}{18}\left[\tan 0 + 4\tan\frac{\pi}{6} + \tan\frac{\pi}{3}\right]$$

$$= \frac{\pi}{18}\left[0 + 4 \times \frac{1}{\sqrt{3}} + \sqrt{3}\right]$$

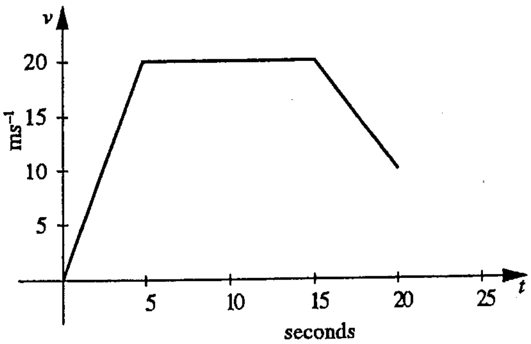
$$= \frac{\pi}{18}\left[\frac{4}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} + \sqrt{3}\right]$$

$$= \frac{\pi}{18}\left(\frac{4\sqrt{3}}{3} + \frac{3\sqrt{3}}{3}\right)$$

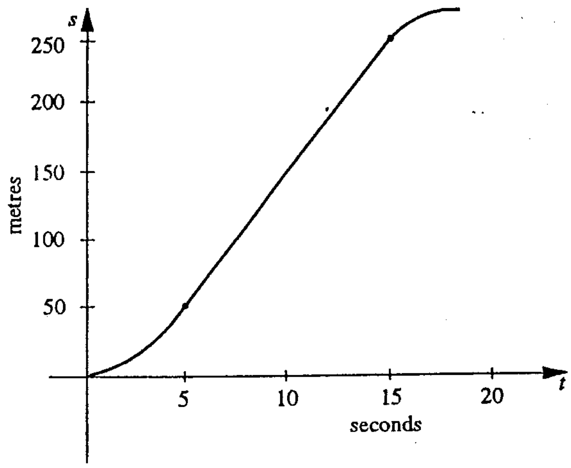
$$= \frac{7\sqrt{3}\pi}{54}$$

This is the change in displacement of the particle between $t = 0$ and $t = \frac{\pi}{3}$.

55. (a)



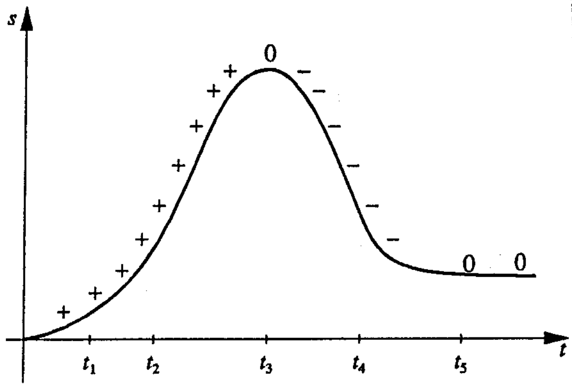
(b)



56. (a) At t_5, the tangent is horizontal, so the velocity is zero.

(b) The tangent is steeper at t_2 than at t_1, so the velocity is greater at t_2 than t_1.

(c)



The velocity is given by the gradient of the tangent along the displacement curve.