+ SOLUTIONS P1

12 3U Test

28.5.01

Begin each section on a new page please.

Section A

[8 marks]

1.

Given that the polynomial $P(x) = 2x^3 - 5x^2 + kx + 15$ has a zero at $x = -3$

(i) determine the value of k.

(ii) factor the polynomial completely.

(iii) sketch the graph of $y = P(x)$, marking all intercepts on the axes clearly.

(iv) state the domain over which $P(x) > 0$.

[8 marks]

2.

If α, β, γ are the roots of $2x^3 - 10x^2 - 8x + 5 = 0$,

(a) evaluate

(i) $\alpha + \beta + \gamma$ (ii) $\alpha\beta + \beta\gamma + \gamma\alpha$ (iii) $\alpha\beta\gamma$

(iv) $\alpha^2 + \beta^2 + \gamma^2$ (v) $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$

(b) write down the equation (with simplest integral coefficients) whose roots are $\alpha^2, \beta^2, \gamma^2$

Section B (New page please)

[9 marks]

3.

A particle moves according to the following rule : $\dfrac{d^2x}{dt^2} = 16(4 - x)$.

(a) Given that $V = 0$ when $x = 8$, show that $V^2 = -16\{(4 - x)^2 - 16\}$.
(b) Establish where the velocity is again zero.
(c) Find the centre of oscillation and period of the motion.
(d) Where is acceleration a maximum?
(e) Graph the motion of the particle.

[7 marks]

4.
(a) Explain why the equation $x^3 - 6x + 1 = 0$ has a root between $x = 2$ and $x = 3$.
(b) By taking the better of these approximations as the first approximation, find, using Newton's Method twice, a third approximation for the root. [Give your answer correct to 2 decimal places.]
(c) Indicate, by checking, whether the third approximation is better than the first.

[OVER]

Section C (New page please)

5. [3 marks]

For the parabola $x = -8p,\ y = -4p^2$

(a) derive the Cartesian form of the equation
(b) write down the equation of the directrix

6. [8 marks]

The chord PQ of a parabola $x^2 = 4ay$ passes through the focus S. The tangents at P and Q meet at T.

(a) Derive the equation of the chord PQ.
(b) Find the relationship between the parameters at P and Q (p, q respectively).
(c) Given the equation of the tangent at P is $y = px - ap^2$ find the point of intersection T of the tangents at P and Q.
(d) Prove that TS is perpendicular to PQ.

7. [5 marks]

The normal to the parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$ on it cuts the axis of the parabola at G.

(a) Derive the equation of this normal.
(b) Find the equation of the locus of the midpoint of PG.

Section D (New page please)

8. [16 marks – 4, 4, 2, 3, 3]

A rocket is fired with a velocity of 90 km/h from the origin, O at an angle of θ to the positive direction of the x - axis. It just misses a bird flying at an altitude of 10 m which is 5 m horizontally from the origin. If we assume that air resistance can be neglected and that the acceleration due to gravity is $g = 10ms^{-2}$:

(a) Prove that the parametric equations describing the particle's motion are :

$$x = 25t\cos\theta \quad \text{and} \quad y = 25t\sin\theta - 5t^2$$

(b) Prove that θ satisfies the equation $\tan^2\theta - 25\tan\theta + 51 = 0$

(c) Find the value(s) of θ to the nearest minute.

(d) The shortest time for the particle to reach the horizontal plane in seconds to 2 decimal places.

(e) The maximum height reached by the projectile to the nearest m.

1 (i) $P(-3) = 0 \quad \therefore \quad -54 - 45 - 3h + 15 = 0$

$$-3h = +84$$

$$h = -28$$

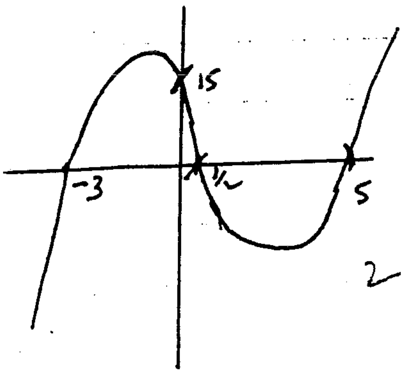
2

(ii) $2x^3 - 5x^2 - 28x + 15 = (x+3)(2x^2 - 11x + 5)$

$$= (x+3)(x-5)(2x-1)$$

3

(iii)



2

(iv) $P(x) > 0$

1

if $-3 < x < \frac{1}{2}$ or $x > 5$

8

2 (a)

$2x^3 - 10x^2 - 8x + 5 = 0$

i.e. $x^3 - 5x^2 - 4x + 5/2 = 0$

(i) $\alpha + \beta + \gamma = 5$

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha = -4$

(iii) $\alpha\beta\gamma = -5/2$

(iv) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$$= 25 - 2(-4)$$

$$= 33$$

2

(v) $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$

$$= 16 - 2 \times (-5/2) \times 5$$

$$= 16 + 25$$

$$= 41$$

2

(b) $\alpha^2\beta^2\gamma^2 = \frac{25}{4}$ $\quad \therefore$ Equation is $x^3 - 33x^2 + 41x - \frac{25}{4} = 0$

1

i.e. $4x^3 - 132x^2 + 164x - 25 = 0$

5

TEST

Y12 3 unit 28/5/2001

q3 [8 marks]

A particle moves according to the following rule $\frac{d^2x}{dt^2} = 16(4-x)$.

(i) Show than $V^2 = -16\{(4-x)^2 - 16\}$ if V=0 when x=8.

(ii) Establish where the velocity is again zero.

(iii) Find the centre of oscillation and period of the motion.

(iv) Where is acceleration a maximum.

(v) Graph the motion of the particle.

$\frac{d^2x}{dt^2} = \frac{d(\frac{1}{2}v^2)}{dx} = 16(4-x)$

alternative treatment through let $-X = (4-x)$ etc ie $\frac{d(\frac{1}{2}v^2)}{dx} = -16X$ etc

a $\Rightarrow \frac{1}{2}v^2 = -\frac{16}{2}(4-x)^2 + C$ or $V^2 = -16(4-x)^2 + C_1$

But when $x=8$ $v=0 \Rightarrow C_1 = 16 \times 16$

$\therefore V^2 = -16(4-x)^2 + 16 \times 16$

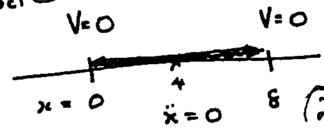
$= -16[(4-x)^2 - 16]$ (2)

b $V=0$ when $x=0$ (1)

c For centre of oscillation $\frac{d^2x}{dt^2} = 0 \Rightarrow x = 4$ (1)

For period $n^2 = 16$ $\therefore T = \frac{2\pi}{n}$ or $\frac{\pi}{2}$ is period (1)

d Accel max at $x=0$ & $x=8$ (2)



(2)

q4 [8 marks]

The equation $x^3 - 6x + 1 = 0$ is known to have a root between $x = 2$ and $x = 3$.

Find, using Newton's Method twice, a better approximation for the root. Is it a better approximation for the root?

let $P(x) = x^3 - 6x + 1$

$P(2) = 8 - 12 + 1 = -3$ & $P(3) = 27 - 18 + 1 = 10$

$\therefore x=2$ is closer to zero of polynomial (1)

— Why? —

$P'(x) = 3x^2 - 6$

let first approximation to zero be $x_0 = 2$

$x_1 \doteqdot x_0 - \frac{P(x_0)}{P'(x_0)}$ (2)

$\therefore x_1 \doteqdot 2 - \left(\frac{-3}{6}\right)$ or $2\frac{1}{2}$

$x_2 \doteqdot 2\frac{1}{2} - \frac{1.625}{12.75}$ or $x_2 \doteqdot 2.5 - 0.1275$

$\doteqdot 2.3725$ (2)

Which is the best approximation of $x_0 = 2$, $x_1 = 2\frac{1}{2}$, $x_2 \doteqdot 2.37$

$P(x_0) = -3$; $P(x_1) = 1.625$ $P(x_2) = 0.09$

We are look for "approximation" Symbol is $\doteqdot$ or $\approx$ not equality

$\therefore x_2 \doteqdot 2.37$ is a better approximation for the zero of $P(x)$. (2)

Check all values, the 3rd may jump away from 2nd or 1st

$\checkmark$ = 1 MARK

5

a) ② $x = -8p$

$\therefore p = \frac{x}{-8}$

$y = -4p^2$ $\checkmark$

$= -4\left(\frac{x}{-8}\right)^2$

$= -\cancel{4}^{1} \times \frac{x^2}{\cancel{64}\,16}$

$x^2 = -16y$ $\checkmark$

(b) ① Directrix is

$y = 4$ $\checkmark$

6 a) ② Let P be $(2ap, ap^2)$

Q $(2aq, aq^2)$

$M_{PQ} = \frac{a(p^2-q^2)}{2ap-2aq}$

$= \frac{\cancel{a}\cancel{(p-q)}(p+q)}{2\cancel{a}\cancel{(p-q)}}$

$= \frac{p+q}{2}$ $\checkmark$

Eqn of PQ is

$y - ap^2 = \frac{p+q}{2}(x-2ap)$

$y = \left(\frac{p+q}{2}\right)x + \cancel{ap^2} - \cancel{ap^2} - apq$

$y = \left(\frac{p+q}{2}\right)x - apq$ $\checkmark$

b) ① $(0, a)$

$a = -apq$ $\checkmark$

$\therefore pq = -1$ $\checkmark$

c) ① $y = px - ap^2$ ①

$y = qx - aq^2$ ②

① − ②

$(p-q)x = a(p^2-q^2)$

$x = a(p+q)$ $\checkmark$

subst into ①

$y = p(a(p+q)) - ap^2$

$= ap^2 + apq - ap^2$

$= apq = -a$ From (b) $(pq=-1)$ $\checkmark$

$\therefore T$ is $(a(p+q), apq)$ or use $(a(p+q), -a)$

d) ③ S $(0, a)$

$M_{TS} = \frac{apq - a}{a(p+q) - 0}$

$= \frac{\cancel{a}(pq-1)}{\cancel{a}\,p+q}$ $\checkmark$

Since $pq = -1$ $= \frac{-1-1}{p+q} = \frac{-2}{p+q}$ $\checkmark$

$M_{PQ} = \frac{p+q}{2}$

$M_{TS} \times M_{PQ} = \frac{-2}{p+q} \times \frac{p+q}{2}$

$= -1$ $\checkmark$

7 i) $x^2 = 4ay$

$\therefore y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$

$= \frac{x}{2a}$

at $x = 2ap$

$m_T = \frac{2ap}{2a}$

$= p$ $\checkmark$

$\therefore M_N = -\frac{1}{p}$

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py - ap^3 - 2ap = 0$ $\checkmark$

FOR G

$x = 0$

$y = \frac{ap^3 + 2ap}{p}$

$= ap^2 + 2a$

(ii) MIDPOINT of PG $\begin{cases} P(2ap, ap^2) \\ G(0, ap^2+2a) \end{cases}$

$= \left(\frac{2ap+0}{2}, \frac{ap^2 + ap^2 + 2a}{2}\right)$

$= (ap, ap^2 + a)$

$x = ap$ ie $p = \frac{x}{a}$

$y = ap^2 + a$

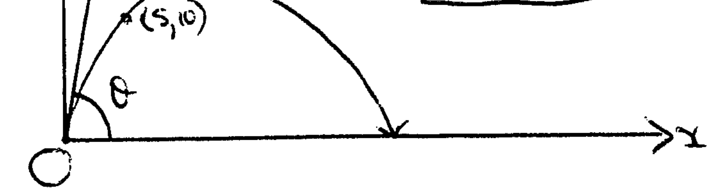
$y = \cancel{a}\frac{x^2}{a^{\cancel{2}}} + a$ $\checkmark$

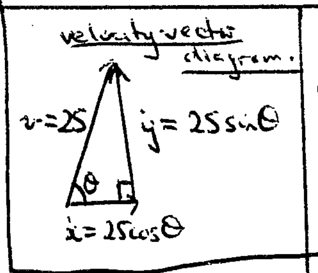
$y = \frac{x^2}{a} + a$ or $x^2 = a(y-a)$

6)

YR 12 3UMT ASSESSMENT 28-5-01

90km/h = 25m/s





i) Initially $\ddot{x} = 0$, $\ddot{y} = -g$

$\therefore \dot{x} = c_1$, $\dot{y} = -gt + c_2$

when $t=0$, $\dot{x} = 25\cos\theta$, $\dot{y} = 25\sin\theta$

$\therefore 25\cos\theta = c_1$, $25\sin\theta = c_2$

$\therefore \dot{x} = 25\cos\theta$, $\dot{y} = 25\sin\theta - gt$

$\therefore x = 25t\cos\theta + c_3$, $y = 25t\sin\theta - \frac{gt^2}{2} + c_4$

when $t=0$, $x=0$, $y=0$.

$\therefore c_3 = 0$, $c_4 = 0$

$\therefore x = 25t\cos\theta$, $y = 25t\sin\theta - 5t^2$
(as $g = 10$).

ie the parametric equations describing the particle's motion are;
$x = 25t\cos\theta$ and $y = 25t\sin\theta - 5t^2$.

(ii) Now coordinates of bird are $(5, 10)$

$\therefore 5 = 25t\cos\theta$ $\quad \therefore t = \frac{\sec\theta}{5}$ — ①

and $10 = 25t\sin\theta - 5t^2$ — ②

sub ① into ②:

$\therefore 10 = 25\left(\frac{\sec\theta}{5}\right)\sin\theta - 5\left(\frac{\sec\theta}{5}\right)^2$

$\therefore 10 = 5\tan\theta - \frac{\sec^2\theta}{5}$

$\therefore 50 = 25\tan\theta - (1+\tan^2\theta)$

$\therefore \tan^2\theta - 25\tan\theta + 51 = 0$

(iii) Now $\tan\theta = \frac{25 \pm \sqrt{25^2 - 4.1.51}}{2}$

$= \frac{25 \pm \sqrt{421}}{2}$

$\therefore \tan\theta = 22.75914226\ldots$
or $2.240857736\ldots$

$\therefore \angle\theta = 87°29'$ or $65°57'$
(to nearest minute)

(iv) Now rocket hits the ground when $y = 0$ $\therefore 0 = 25t\sin\theta - 5t^2$

$\therefore 5t(5\sin\theta - t) = 0$

$\therefore \underbrace{t = 0}_{\text{initial}}$ or $\underbrace{t = 5\sin\theta}_{\text{final}}$

$\therefore$ shortest time taken will be when $\angle\theta = 65°57'$ (approx)

$\therefore t = 5\sin\theta$

$\therefore t = 4.565979801\ldots$

$\therefore$ time taken is 4.57 s (2 d.p.)

(v) For greatest height $\dot{y} = 0$

$\therefore 0 = 25\sin\theta - 10t$

$\therefore t = \frac{25\sin\theta}{10} = \frac{5\sin\theta}{2}$

$\therefore y_{max} = 25\left(\frac{5\sin\theta}{2}\right)\sin\theta - 5\left(\frac{5\sin\theta}{2}\right)^2$

$= \frac{125}{2}\sin^2\theta - \frac{125\sin^2\theta}{4}$

$= \frac{125\sin^2\theta}{4}$

Now for greatest height $\angle\theta \doteqdot 87°29'$

$\therefore$ greatest height $= 31.18978557$.

$= 31$ m (to neares m