CRANBROOK SCHOOL

YEAR 12 EXT1-11ACC MATHEMATICS – TEST

14^{th} March, 2006 Circle teacher: SKB JJA CJL

- Geometrical Applications of Calculus
- Approximation to roots of $P(x)=0$
- Parametrics
- Integration

Time: 50mins

All necessary working should be shown in every question.
Full marks may not be awarded if work is careless or badly arranged.
Approved silent calculators may be used.
Begin each question on a new page.

1. **(12 marks) (Begin a new page)** SKB

(a) A closed cylindrical can contains $2156\,cm^3$ of air.

(i) Show that the surface area can be expressed as:
$SA=\dfrac{4312}{r}+2\pi r^2$. 2

(ii) Hence calculate its minimum surface area to the nearest cm^2. 3

(b) (i) Use long division to show that the curve $y=\dfrac{2x^2}{x-1}$ can be expressed as
$y=2x+2+\dfrac{2}{x-1}$. 1

(ii) Hence or otherwise determine the oblique asymptote, any other asymptotes, any stationary points and any points of inflexion. 5

(iii) Hence sketch the curve $y=\dfrac{2x^2}{x-1}$, showing these features. 1

2. **(12 marks) (Begin a new page)** JJA

(a) By using the 'halving the interval' method twice find an approximation to the root of $x^3+2x-8=0$ in the interval $1<x<2$. 3

(b) If $P(x) = 2x^3 - 3x^2 + 1.1$ has a root near $x = 1$. By using $z_1 = 0.4$ as a first approximation find a closer approximation to this root by using Newton's Method once leaving your answer correct to 3 decimal places. 3

Why would Newton's Method have failed if $z_1 = 1$ had been chosen as the first approximation? 1

(c) (i) Show that tangents to the parabola $x^2 = 4ay$ at the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ intersect at the point $T(a(p+q), apq)$. 2

(ii) If $p^2 + q^2 = 2$ find the equation of the locus of T. Show that this locus is a parabola. 3

3. **(12 marks)** **(Begin a new page)** **CJL**

(a) Use Simpson's Rule with 6 subintervals to find an approximation to the area bounded by the curve $y = \dfrac{1}{x^3 - 1}$, the x-axis and lines $x = 2$ and $x = 5$ correct to 4 decimal places. 4

(b) Find the exact volume generated when the area between the line $y = x + 2$ and the curve $y = x^2$ is rotated about the x-axis. 4

(c) By using the substitution $u^2 = 3x + 4$ evaluate $\int_0^4 x\sqrt{3x+4}\ dx$. 4

1) (a)(i) $2156 = \pi r^2 h$ —— ①

$SA = 2\pi r h + 2\pi r^2$ —— ②

from ① $h = \dfrac{2156}{\pi r^2}$ sub into ②

$\therefore SA = 2\pi r\left(\dfrac{2156}{\pi r^2}\right) + 2\pi r^2$

$= \dfrac{4312}{r} + 2\pi r^2$

(ii) $\dfrac{dSA}{dr} = -\dfrac{4312}{r^2} + 4\pi r$

$\dfrac{d^2SA}{dr^2} = \dfrac{8624}{r^3} + 4\pi$

For a possible max/min $\dfrac{dSA}{dr} = 0$

$\therefore \dfrac{4312}{r^2} = 4\pi r$

$\therefore r^3 = \dfrac{4312}{4\pi} = \dfrac{1078}{\pi}$

$\therefore r = \sqrt[3]{\dfrac{1078}{\pi}}$

when $r = \sqrt[3]{\dfrac{1078}{\pi}}$ $\dfrac{d^2SA}{dr^2} > 0 \Rightarrow$ min. surface area when $r = \sqrt[3]{\dfrac{1078}{\pi}}$

This minimum surface area $= 923.876\ldots$

$= 924\ cm^2$ (to nearest cm^2)

(b)(i) $y = \dfrac{2x^2}{x-1}$

$$\begin{array}{r}
2x+2 \\
x-1\,\overline{)\,2x^2} \\
-(2x^2-2x) \\ \hline
2x \\
-(2x-2) \\ \hline
2
\end{array}$$

$\therefore y = 2x+2 + \dfrac{2}{x-1}$

(ii) As $x \to \pm\infty$ $y \to 2x+2$

$\Rightarrow$ oblique asymptote at $y = 2x+2$.

y is undefined when $x=1 \Rightarrow$ vertical asymptote at $x=1$

when $x=0$ $y=0$ $\Rightarrow$ intercept at $(0,0)$.

$y = \dfrac{2x^2}{x-1}$

$\therefore y' = \dfrac{(x-1).4x - 2x^2.1}{(x-1)^2}$

$= \dfrac{4x^2-4x-2x^2}{(x-1)^2}$

$= \dfrac{2x^2-4x}{(x-1)^2}$

$= \dfrac{2x(x-2)}{(x-1)^2}$

$y'' = \dfrac{(x-1)^2.(4x-4) - (2x^2-4x).2(x-1)}{(x-1)^4}$

$= \dfrac{2(x-1)[(x-1).(2x-2) - (2x^2-4x)]}{(x-1)^4}$

$= \dfrac{2[2x^2-4x+2-2x^2+4x]}{(x-1)^3}$

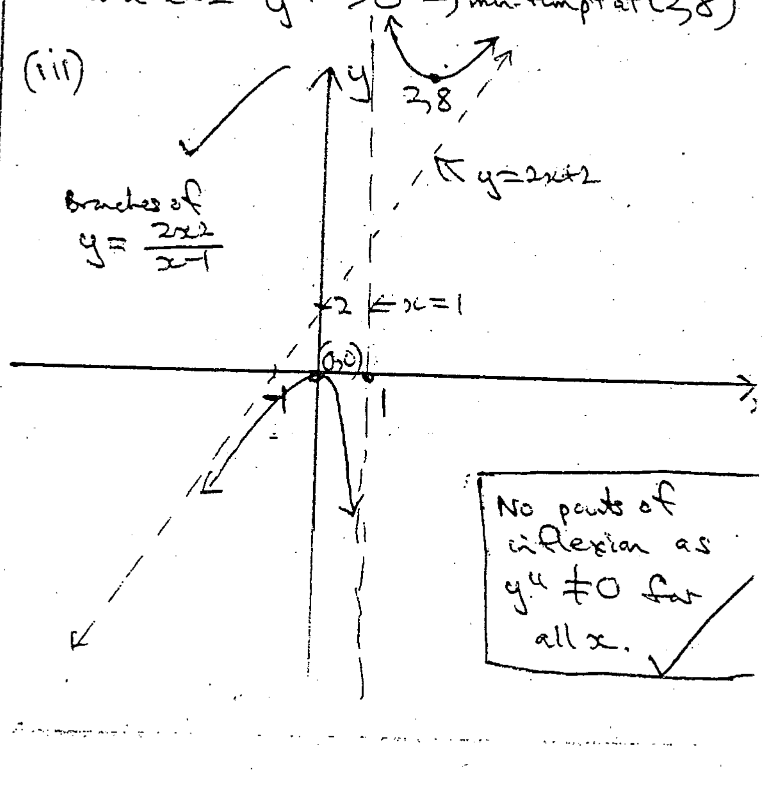
$= \dfrac{4}{(x-1)^3}$

For a stat. pt $y'=0$ $\therefore x=0$ or 2

when $x=0$ $y''<0 \Rightarrow$ max. turn pt at $(0,0)$

when $x=2$ $y''>0 \Rightarrow$ min. turn pt at $(2,8)$

(iii)



No points of inflexion as $y'' \neq 0$ for all x.

2 (a) Let $P(x) = x^3+2x-8$

$P(1) = -5 < 0$
$P(2) = 4 > 0$ } as $P(1)$ and $P(2)$ have opposite signs

and $P(x)$ is cts for all x.

$\Rightarrow$ there is at least 1 real root in the interval $1<x<2$.

Let $x_1 = \frac{1+2}{2} = 1.5$

$P(1.5) = -1.625 < 0$

$\therefore$ as $P(1.5)$ and $P(2)$ have opposite signs

$\Rightarrow$ at least 1 real root in interval $1.5<x<2$

$\therefore x_2 = \frac{1.5+2}{2} = 1.75$

$\therefore$ Approx to root of $P(x)=0$ is $x = 1.75$ after using the 'halving the interval' method twice.

(b) $P(x) = 2x^3-3x^2+1.1$

$P'(x) = 6x^2-6x$

$z_1 = 0.4$

By Newton's method $z_2 = z_1 - \frac{P(z_1)}{P'(z_1)}$

$\therefore z_2 = 0.4 - \frac{P(0.4)}{P'(0.4)}$

$= 0.4 - \frac{0.748}{-1.44}$

$= 0.919$ (3dp)

At $z_1=1$ $P'(1) = 0$

$\Rightarrow$ there is a stationary point on the curve at $x=1$. A tangent drawn to the curve at this point will be parallel to the x-axis and $\therefore$ not give a 2nd approx to the root.

$\therefore$ Newton's method will fail (z_2 will be undefined)

(c) (i) $x^2 = 4ay$ $\therefore y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$

At $P(2ap, ap^2)$ $\frac{dy}{dx} = p = m_{\text{tangent}}$

$\therefore$ Eqn of tangent at P is:

$y - ap^2 = p(x-2ap)$

$\therefore y - ap^2 = px - 2ap^2$

$\therefore y = px - ap^2$ —— ①

Similarly, the eqn of the tangent at Q is:

$y = qx - aq^2$ —— ②

① − ②: $0 = x(p-q) - a(p^2-q^2)$

$\therefore x = \frac{a(p-q)(p+q)}{(p-q)} = a(p+q)$

sub $x = a(p+q)$ into ①

$\therefore y = pa(p+q) - ap^2$

$\therefore y = ap^2 + apq - ap^2 = apq$

$\Rightarrow$ point of intersection T is: $(a(p+q), apq)$

(ii) Now from T: $x = a(p+q)$
and $y = apq$

$\therefore p+q = \frac{x}{a}$ —— ①

$pq = \frac{y}{a}$ —— ②

Now $(p+q)^2 = p^2+q^2+2pq$

$\therefore \left(\frac{x}{a}\right)^2 = 2 + 2\left(\frac{y}{a}\right)$

$\therefore \frac{x^2}{a^2} = 2\left(1+\frac{y}{a}\right)$

$\therefore x^2 = 2a^2\left(1+\frac{y}{a}\right)$

$x^2 = 2a^2 + 2ay$

$\therefore \underline{x^2 = 2a(y+a)}$ is the locus!

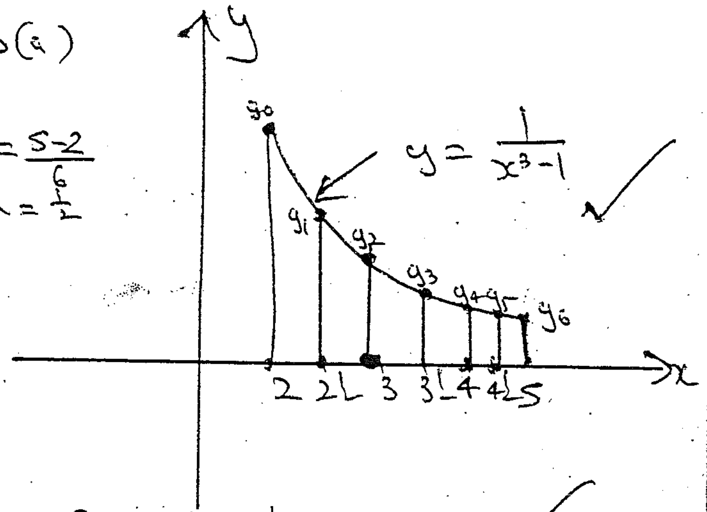
which is of the form $x^2 = 4a(y-k)$

$\therefore$ locus is a parabola.

3(a)

$h = \frac{5-2}{6}$

$\therefore h = \frac{1}{2}$



By Simpson's Rule,

$$\text{Area} \doteq \frac{h}{3}\left[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$\therefore \text{Area} = \frac{1}{6}\left[\frac{1}{7} + \frac{1}{124} + 4\left(\frac{1}{14.625} + \frac{1}{41.875} + \frac{1}{90.125}\right) + 2\left(\frac{1}{28} + \frac{1}{63}\right)\right]$$

$= 0.111250954\ldots$

$= 0.1113$ units^2 (4 d.p.)

(b) $y = x+2$ — ①

$y = x^2$ — ②

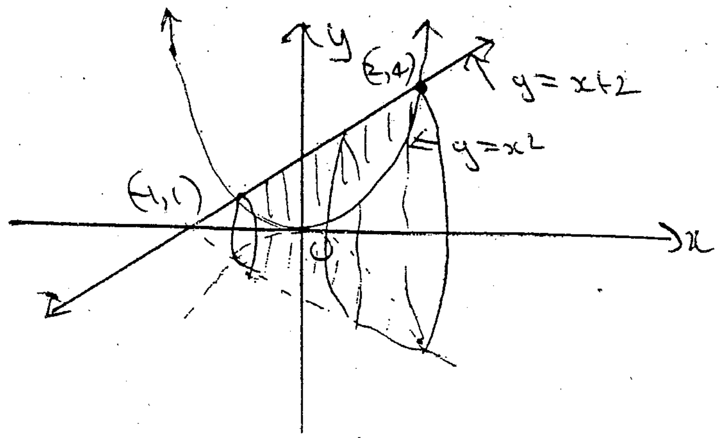
② − ①: $0 = x^2 - x - 2$

$\therefore 0 = (x-2)(x+1)$

$\therefore x = 2$ or -1

when $x=2$ $y=4$, when $x=-1$ $y=1$

$\Rightarrow$ pts of int at $(2,4)$ and $(-1,1)$



$$\text{Volume} = \pi\int_{-1}^{2} (x+2)^2 - (x^2)^2\,dx$$

$$= \pi\int_{-1}^{2} x^2 + 4x + 4 - x^4\,dx$$

$$= \pi\left[\frac{x^3}{3} + 2x^2 + 4x - \frac{x^5}{5}\right]_{-1}^{2}$$

$$= \pi\left[\left(\frac{8}{3} + 8 + 8 - \frac{32}{5}\right) - \left(-\frac{1}{3} + 2 - 4 + \frac{1}{5}\right)\right]$$

$$= \frac{72\pi}{5} \text{ units}^3$$

(c) $I = \int_0^4 x\sqrt{3x+4}\,dx$

let $u^2 = 3x+4$, $x = \frac{u^2-4}{3}$

$\therefore 2u\frac{du}{dx} = 3$

$\therefore \frac{2u}{3}\,du = dx$

when $x=0$ $u=2$

$x=4$ $u=4$

$$\therefore I = \int_2^4 \left(\frac{u^2-4}{3}\right) u \cdot \frac{2u}{3}\,du$$

$$= \frac{2}{9}\int_2^4 u^4 - 4u^2\,du$$

$$= \frac{2}{9}\left[\frac{u^5}{5} - \frac{4u^3}{3}\right]_2^4$$

$$= \frac{2}{9}\left[\left(\frac{1024}{5} - \frac{256}{3}\right) - \left(\frac{32}{5} - \frac{32}{3}\right)\right]$$

$$= 27\frac{67}{135}$$