INDUCTION – WORKSHEET

COURSE/LEVEL

NSW Secondary High School Year 11 Preliminary Mathematics Extension. Syllabus reference: 7.4.

1. Prove the following by induction, where n is any positive integer:

(a) $1+2+3+\ldots+n=\dfrac{n(n+1)}{2}$.

(b) $1^2+2^2+3^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$.

(c) $1^3+2^3+3^3+\ldots+n^3=\dfrac{n^2(n+1)^2}{4}$.

* (d) $1^3+2^3+3^3+\ldots+n^3=(1+2+3+\ldots n)^2$.

(e) $1+3+5+7+\ldots+(2n-1)=n^2$.

(f) $\dfrac{1}{1.2}+\dfrac{1}{2.3}+\dfrac{1}{3.4}+\ldots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$.

(g) $\dfrac{1}{1.4}+\dfrac{1}{4.7}+\dfrac{1}{7.10}+\ldots+\dfrac{1}{(3n-2)(3n+1)}=\dfrac{n}{3n+1}$.

(h) $1.2.3+2.3.4+3.4.5+\ldots+n(n+1)(n+2)=\dfrac{n(n+1)(n+2)(n+3)}{4}$.

2. Prove the following results for all integers $n\geq 1$.

(a) 3^n+1 is divisible by 2.

(b) $3^{2n}-1$ is divisible by 8.

* (c) $5^n+2(11^n)$ is divisible by 3.

(d) $5^{2n}+5^n+2$ is divisible by 4.

3. Prove that each of the following expressions are divisible by 5 if n is any positive integer.

(a) $2^{3n}-3^n$.

(b) $3^{3n}+2^{n+2}$.

(c) $9^{n+2}-2^{2n}$.

(d) $13(6^n)+2$.

4. Prove the following:

(a) $3^n\geq 1+2n$.

(b) $4^n\geq 1+3n$.

Check corrections!

(1a) Let $S_n = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$

Step 1. Prove true for $n=1$

$1 = \frac{1(1+1)}{2}$

ie. $1 = \frac{2}{2}$ ✓ $\therefore$ true for $n=1$

Step 2. Assume true for $n=k$

ie. $S_k = 1+2+3+\ldots+k = \frac{k(k+1)}{2}$ ✓

Prove true for $n=k+1$

ie. $S_{k+1} = 1+2+3+\ldots+k+(k+1) = \frac{k(k+1)}{2} + (k+1)$ ✓

$= \frac{k(k+1)+2(k+1)}{2}$

$= \frac{(k+1)(k+2)}{2}$ ✓ which is of the form $S_n = \frac{n(n+1)}{2}$ with n replaced by $k+1$.

Step 3. If result is true for $n=k$ and for $n=k+1$, then it is true for all positive integer values of n.

(b) Let $S_n = 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Step 1. Prove true for $n=1$

$1^2 = \frac{1(1+1)(2\times1+1)}{6}$

ie. $1 = \frac{1\times2\times3}{6}$ ✓

Step 2. Assume result is true for $n=k$

ie. $S_k = 1^2+2^2+3^2+\ldots+k^2 = \frac{k(k+1)(2k+1)}{6}$

Prove true for $n=k+1$

ie. $S_{k+1} = 1^2+2^2+3^2+\ldots+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$ ✓

$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$

$= \frac{k+1}{6}\left(k(2k+1)+6(k+1)\right) = \frac{k+1}{6}(2k^2+k+6k+6)$ ✓

$= \frac{k+1}{6}(2k^2+7k+6)$

$= \frac{k+1}{6}(k+2)(2k+3)$ ✓ which is $S_n = \frac{n(n+1)(2n+1)}{6}$ with n replaced by $k+1$.

Step 3. If result is true for $n=k$ & for $n=k+1$, then it is true for all positive integer values of n.

(c) Let $S_n = 1^3+2^3+3^3+\dots+n^3 = \frac{n^2(n+1)^2}{4}$

Step 1. Prove true for $n=1$

$1^3 = \frac{1^2(1+1)^2}{4}$

ie. $1 = \frac{4}{4}$ ✓

Step 2. Assume result is true for $n=k$.

ie. $S_k = 1^3+2^3+3^3+\dots+k^3 = \frac{k^2(k+1)^2}{4}$

Prove true for $n=k+1$

ie. $S_{k+1} = 1^3+2^3+3^3+\dots+k^3+(k+1)^3$

$= \frac{k^2(k+1)^2}{4} + (k+1)^3$

$= \frac{k^2(k+1)^2+4(k+1)^3}{4}$ ✓

$= \frac{(k+1)^2}{4}(k^2+4(k+1)) = \frac{(k+1)^2}{4}(k^2+4k+4)$ ✓

$= \frac{(k+1)^2}{4}(k+2)^2$

$= \frac{(k+1)^2}{4}((k+1)+1)^2$ ✓ which is of the form $S_n = \frac{n^2(n+1)^2}{4}$ ✓

with n replaced by $k+1$.

Step 3.

If result is true for $n=k$ and $n=k+1$, it is also true for all positive integers values of n.

(d) Let $S_n = 1^3+2^3+3^3+\dots+n^3 = (1+2+3+\dots n)^2$

Step 1. Prove true for $n=1$

$1^3 = 1^2$ ✓

Step 2. Assume result is true for $n=k$

ie. $S_k = 1^3+2^3+3^3+\dots+k^3 = (1+2+3+\dots+k)^2$

Prove true for $n=k+1$

ie. $S_{k+1} = 1^3+2^3+3^3+\dots+k^3+(k+1)^3 = (1+2+3+\dots+k)^2+(k+1)^3$

$S_n = \frac{n}{2}(n+1)$

$= \left(\frac{k}{2}(1+k)\right)^2 + (k+1)(k+1)^2$ ✓ $= (k+1)^2\left(\frac{k^2}{4}+k+1\right)$ ✓ $= (k+1)^2\left(\frac{k^2+4k+4}{4}\right)$

$= (k+1)^2\frac{(k+2)^2}{4} = \frac{(k+1)^2(k+2)^2}{4} = \frac{(k+1)^2}{4}(k+2)^2$ ✓ ✓

now, S_n of $(1+2+3+\dots+k+(k+1))^2 = \left(\frac{k+1}{2}(1+k+1)\right)^2 = \left(\frac{k+1}{2}\right)^2(k+2)^2 = \frac{(k+1)^2(k+2)^2}{4}$

Step 3. If result is true for $n=k$, and $n=k+1$, it is also true for all positive integer values of n.

(e) Let $S_n = 1+3+5+7+\ldots+(2n-1) = n^2$

Step 1. Prove true for $n=1$

ie. $1 = 1^2$ ✓

Step 2. Assume result is true for $n=k$.

ie. $S_k = 1+3+5+7+\ldots+(2k-1) = k^2$

Prove true for $n=k+1$.

ie. $S_{k+1} = 1+3+5+7+\ldots+(2k-1)+(2k+1) = k^2+(2k+1)$

$= k^2+2k+1$ ✓

$= (k+1)^2$ ✓ which is $S_n = n^2$ with n replaced by $k+1$.

Step 3. If result is true for $n=k$ and $n=k+1$, it is also true for all positive integer values of n.

(f) Let $S_n = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Step 1. Prove true for $n=1$

ie. $\frac{1}{1\times 2} = \frac{1}{1+1}$ ✓

Step 2. Assume result is true for $n=k$.

ie. $S_k = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

Prove true for $n=k+1$.

ie. $S_{k+1} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$ ✓

$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)}$ ✓

$= \frac{(k+1)^{\cancel{2}}}{\cancel{(k+1)}(k+2)} = \frac{k+1}{k+2}$ ✓

$= \frac{k+1}{(k+1)+1}$ ✓ which is $S_n = \frac{n}{n+1}$ with n replaced by $k+1$.

Step 3. If result is true for $n=k$ and $n=k+1$, then it is true for all positive integer values of n.

(g) Let $S_n = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$

Step 1. Prove true for $n=1$.

$\frac{1}{1\times 4} = \frac{1}{3\times 1+1}$ ✓

Step 2. Assume result is true for $n=k$

ie. $S_k = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}$

→

Prove true for $n=k+1$

ie. $S_{k+1} = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3k+1)(3k+4)}$ ✓

$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$ ✓

$= \frac{k(3k+4)+1}{(3k+1)(3k+4)} = \frac{3k^2+4k+1}{(3k+1)(3k+4)}$

$= \frac{\cancel{(3k+1)}(k+1)}{\cancel{(3k+1)}(3k+4)} = \frac{k+1}{3k+4}$ ✓

$= \frac{k+1}{3(k+1)+1}$ ✓ which is $S_n = \frac{n}{3n+1}$ with n replaced by $k+1$.

(h) Let $S_n = 1.2.3 + 2.3.4 + 3.4.5 + \dots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

Step 1. Prove true for $n=1$

$1 \times 2 \times 3 = \frac{1(1+1)(1+2)(1+3)}{4}$

ie. $6 = \frac{2 \times 3 \times 4}{4} = \frac{24}{4}$ ✓

Step 2. Assume result is true for $n=k$.

ie. $S_k = 1.2.3 + 2.3.4 + 3.4.5 + \dots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}$

Prove true for $n=k+1$.

ie. $S_{k+1} = 1.2.3 + 2.3.4 + 3.4.5 + \dots + k(k+1)(k+2) + (k+1)(k+2)(k+3)$

$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$ ✓

$= \frac{k(k+1)(k+2)(k+3) + 4(k+1)(k+2)(k+3)}{4}$

$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$ ✓

$= \frac{(k+1)((k+1)+1)((k+1)+2)((k+1)+3)}{4}$ ✓ which is $S_n = \frac{n(n+1)(n+2)(n+3)}{4}$

with n replaced by $k+1$.

(2a) 3^n+1 divisible by 2 $(n \geq 1)$

Step 1 Prove true for $n=1$

ie. $3^1+1=4$ ✓

Step 2 Assume result is true for $n=k$

ie. $3^k+1 = 2N$ (N is an integer)

Prove true for $n=k+1$

$3^{k+1}+1 = 3^k \times 3 + 1$

$= 3(3^k)+1$

$= 3(2N-1)+1$ ✓ (from $3^k+1=2N$)

$= 6N-2$

$= 2(3N-1)$ ($3N-1$ is an integer since N is an integer)

$= 2Q$ ✓

Step 3 Since result is true for $n=k$ and $n=k+1$, it is true for all integer values of n greater than or equal to 1.

(2b) $3^{2n}-1$ divisible by 8 ($n \geq 1$)

Step 1 Prove true for $n=1$

ie. $3^{2n}-1 = 8$ ✓

Step 2 Let $n=k$ be value for which result holds

ie. $3^{2k}-1 = 8N$ (N is an integer)

Prove true for $n=k+1$

ie $3^{2k+2}-1 = 3^{2k} \times 3^2 - 1$

$= 9(3^{2k})-1$ ✓

$= 9(8N+1)-1$ (from $3^{2k}-1=8N$)

$= 72N+8$

$= 8(9N+1)$ ✓ ($9N+1$ is an integer since N is an integer)

$= 8Q$ ✓

Step 3. If result holds for $n=k$, and $n=k+1$, then it also holds for all integer values of n greater than or equal to 1.

(c) $5^n+2(11^n)$ divisible by 3 ($n \geq 1$)

Step 1. Prove true for $n=1$

$5+2\times 11 = 27$ ✓

Step 2 Assume result is true for $n=k$

ie. $5^k+2(11^k) = 3N$ (N is an integer)

Prove true for $n=k+1$

ie. $5^{k+1}+2(11^{k+1}) = 5(5^k)+2(11(11^k))$

$= 5(5^k)+22(11^k)$ ✓

~~$= 1.5^k + 2.11^k + 4.5^k + 20.11^k$~~

$= 5[1.5^k+2.11^k] + 12.11^k$ ✓

Result is true for $n=k$, and $n=k+1$. Hence, it is true for all integer values of n greater than 1.

$= 5 \times 3N + 12.11^k$

$= 15N + 12.11^k$

$= 3(5N+4.11^k)$ ✓ ($5N+4.11^k$ is an integer since N is an integer)

$= 3Q$ ✓ Step 3.

(d) $5^{2n}+5^n+2$ is divisible by 4

Step 1. Prove true for $n=1$

ie. $5^2+5^1+2 = 32$ ✓

Step 2. Assume result is true for $n=k$.

ie. $5^{2k}+5^k+2 = 4N$ (N is an integer)

Prove true for $n=k+1$

ie. $5^{2k+3}+5^{k+1}+2 \ldots$ ✓

$= 5^{2k}\times 5^3 + 5^k \times 5 + 2 = 5^3(5^{2k})+5(5^k)+2$

$= 125(5^{2k})+5(5^k)+2$

$= 5(5^{2k}+5^k+2)+120.5^{2k}-8$

$= 5\times 4N + 120\times 5^{2k} - 8$ ✓

$= 20N + 120\times 5^{2k} - 8$

$= 4(5N+30.5^{2k}-2)$

$= 4Q$ ✓ (Q is an integer)

Step 3. If result is true for $n=k$ and $n=k+1$, then it is true for all integer values of n greater than 1.

(3a) $2^{3n}-3^n$ divisible by 5 ($n \geq 1$)

Step 1 Prove true for $n=1$

ie. $2^3-3 = 5$ ✓

Step 2. Assume result is true for $n=k$.

ie. $2^{3k}-3^k = 5N$ (N is an integer)

Prove true for $n=k+1$

ie. $2^{3k+3} - 3^{k+1}$ ✓

$= 2^{3k} \times 2^3 - 3^k \times 3 = 8(2^{3k}) - 3(3^k)$

$= 3(2^{3k} - 3^k) + 5(2^{3k})$

$= 3 \times 5N + 5(2^{3k})$

$= 15N + 5(2^{3k}) = 5(3N + 2^{3k}) = 3Q$

(Q is another integer)

Step 3. If result is true for $n=k$ & $n=k+1$, it is also true for all positive integer values of n.

(3b) $3^{3n} + 2^{n+2}$ is divisible by 5.

Step 1. Prove true for $n=1$

ie. $3^3 + 2^3 = 35$ ✓

Step 2. Assume result is true for $n=k$

ie. $3^{3k} + 2^{k+2} = 5N$ (N is an integer)

Prove true for $n=k+1$

ie. $3^{3k+3} + 2^{k+3}$ ✓ $= 27(3^{3k}) + 2(2^{k+2})$

$= 2(3^{3k} + 2^{k+2}) + 25 \times 3^{3k}$ ✓

$= 2 \times 5N + 25.3^{3k} = 10N + 25.3^{3k}$ ✓

$= 5(2N + 5.3^{3k})$ ✓ $= \cancel{3Q}\ 5Q$

(Q is another integer).

(3c) $9^{n+2} - 2^{2n}$ divisible by 5

Step 1. Prove true for $n=1$.

ie. $9^3 - 2^2 = 725$ ✓

Step 2. Assume result is true for $n=k$

ie. $9^{k+2} - 2^{2k} = 5N$ (N is an integer)

Prove true for $n=k+1$

ie. $9^{k+3} - 2^{2k+2} = 9(9^{k+2}) - 4(2^{2k})$

$= 4(9^{k+2} - 2^{2k}) + 5.9^{k+2}$ ✓

$= 4 \times 5N + 5.9^{k+2}$

$= 5(4N + 9^{k+2})$ ✓ $= 5Q$ ✓ (Q is an integer)

Step 3. Result is true for $n=k$ and $n=k+1$, therefore it is true for all positive integer values of n.

(3d) $13(6^n) + 2$ divisible by 5

Step 1. Prove true for $n=1$

ie. $13 \times 6^1 + 2 = 80$ ✓

Step 2. Assume result is true for $n=k$.

ie. $13(6^k) + 2 = 5N$ (N is integral)

Prove true for $n=k+1$

ie. $13(6^{k+1}) + 2 = 13 \times 6(6^k) + 2$

$= 78(6^k) + 2$ ✓

$= 13(6^k) + 2 + 65(6^k)$ ✓

$= 5N + 65(6^k) = 5(N + 13.6^k) = 5Q$ ✓

(Q is another integer).

Step 3. If result is true for $n=k$, and $n=k+1$, then it is also true for all positive integer values of n.

(4a) $3^n \geq 1 + 2n$ $(n \geq 1)$

Step 1. Prove true for $n=1$

ie. $3^1 \geq 1 + 2$ ✓

Step 2. Assume result is true for $n=k$.

ie. $3^k \geq 1 + 2k$ $(k \geq 1)$

ie. $3^k - 1 - 2k \geq 0$ ✓

Prove true for $n=k+1$

ie. $3^{k+1} - 1 - 2(k+1) \geq 0$ ✓

ie. $3(3^k) - 2k - 3 \geq 0$

LHS $> 3(1+2k) - 2k - 3$ (from $3^k \geq 1 + 2k$)

$= 3 + 6k - 2k - 3$

$= 4k$ ✓

> 0, since $k \geq 1$

Step 3. Result is true for $n=k$ and $n=k+1$, therefore it is true for all positive integer values of n.

4.b) $4^n \geq 1 + 3n \quad (n \geq 1)$

Step 1. Prove true for $n=1$

ie. $4 \geq 1+3$ ✓

Step 2. Assume result is true for $n=k$

ie. $4^k \geq 1+3k \Rightarrow 4^k - 1 - 3k \geq 0 \quad (k \geq 1)$

Prove true for $n=k+1$

ie. $4^{k+1} - 1 - 3(k+1) \geq 0$ ✓

ie. $4(4^k) - 4 - 3k \geq 0$

$LHS \geq 4(1+3k) - 4 - 3k \quad$ (from $4^k \geq 1+3k$)

$= 4 + 12k - 4 - 3k$ ✓

$= 9k$

≥ 0 (since $k \geq 1$) ✓

Step 3. Result is true for $n=k, n=k+1$, therefore it is true for all positive integer values of n.