Diagnostic Topic Tests: NSW Mathematics Extension 1 (Preliminary & HSC)

Test 4: Inverse Functions

Total 40 marks (Suggested time: 45 minutes)

Directions to students

- Attempt ALL questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- The marks for each question are indicated at the start of the question.

QUESTION 1. (10 marks) Marks

(a) For each of the following functions, find its inverse and sketch the function and its inverse on the same number plane. Clearly label each graph. 8

(i) $f(x) = \frac{5-3x}{2}$

(ii) $g(x) = \sqrt{2x-3}$

(b) Evaluate $\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1}(-1)$ without using the calculator. 2

QUESTION 2. (10 marks)

(a) If $f(x) = x^2 \sin^{-1} x$, show that $f'\left(\frac{1}{2}\right) = \frac{1}{6}(\pi + \sqrt{3})$. 4

(b) Consider the function $y = 3\sin^{-1}(2x)$. 6

(i) State the domain and range.

(ii) Sketch the graph, showing the important features.

(iii) Find $\frac{dy}{dx}$.

(iv) State the values of x for which $\frac{dy}{dx}$ is defined.

QUESTION 3. (10 marks) Marks

(a) Show that $\int_0^{\frac{3}{2}} \frac{dt}{\sqrt{9-2t^2}} = \frac{\pi}{4\sqrt{2}}$. 3

(b) The region bounded by the curve $y = \frac{1}{\sqrt{9+x^2}}$, the lines $x = -3$ and $x = 3$ and the x-axis is rotated about the x-axis. Find the volume of the solid formed. 4

(c) Without using a calculator, show that $\tan^{-1}\left(\frac{2}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) = \frac{\pi}{4}$. 3

QUESTION 4. (10 marks)

(a) Consider the function $y = \cos^{-1}(2x-1)$. 3

(i) State the domain and range.

(ii) Sketch the curve.

(b) Find the exact value of $\sin\left(2\cos^{-1}\frac{15}{17}\right)$. 3

(c) Consider the functions $y = -\cos^{-1} x$ and $y = 2\tan^{-1}(x-1)$. 4

(i) Show that the graphs of these functions intersect on the y-axis.

(ii) Show that the graphs have a common tangent at this point of intersection.

Test 4: Inverse Functions

Suggested Solutions

QUESTION 1.

(a) (i) Let $y = \frac{5-3x}{2}$

Note: Always write in the form $y = f(x)$

Inverse is $x = \frac{5-3y}{2}$

Note: Interchange x and y.

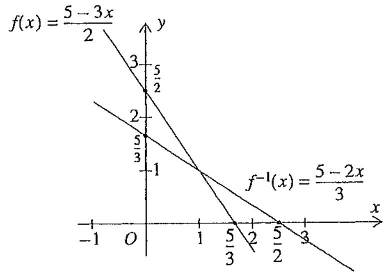
$$2x = 5 - 3y$$
$$3y = 5 - 2x$$
$$y = \frac{5-2x}{3}$$

Note: Solve for y.

$$f^{-1}(x) = \frac{5-2x}{3}$$

Note: Replace y by $f^{-1}(x)$.

Note: The domain and range of both f and f^{-1} consist of all real numbers.



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(ii) Let $y = \sqrt{2x-3}$, $x \geq \frac{3}{2}$, $y \geq 0$.

Note: Always write in the form $y = f(x)$

Inverse is $x = \sqrt{2y-3}$, $y \geq \frac{3}{2}$, $x \geq 0$

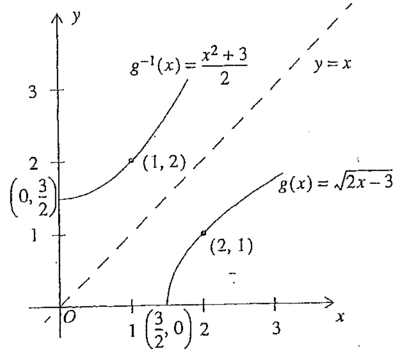
Note: Interchange x and y.

$$x^2 = 2y - 3$$
$$2y = x^2 + 3$$
$$y = \frac{x^2+3}{2}$$

$g^{-1}(x) = \frac{x^2+3}{2}$ for $x \geq 0$.

Note: Replace y by $g^{-1}(x)$

Note: In the original function $y \geq 0$. Therefore $x \geq 0$ in the inverse function.



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Note: The graph of $y = g^{-1}(x)$ is the reflection of the graph of $y = g(x)$ in the line $y = x$.

Note: The domain and range for both the function and its inverse have restrictions as shown in the diagram.

(b)
$$\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1}(-1) = \frac{\pi}{3} + \left(-\frac{\pi}{4}\right)$$
$$= \frac{\pi}{12}$$

Note: $\tan^{-1}(-x) = -\tan^{-1}x$

2

QUESTION 2.

(a) $f(x) = x^2 \sin^{-1} x$

$$f'(x) = 2x \times \sin^{-1} x + \frac{1}{\sqrt{1-x^2}} \times x^2$$

Note: Using product rule.

$$= 2x \sin^{-1} x + \frac{x^2}{\sqrt{1-x^2}}$$

$$f'\left(\frac{1}{2}\right) = 2 \times \frac{1}{2} \times \sin^{-1}\frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{\sqrt{1-\left(\frac{1}{2}\right)^2}}$$

$$= \frac{\pi}{6} + \frac{\frac{1}{4}}{\sqrt{\frac{3}{4}}}$$

$$= \frac{\pi}{6} + \frac{\frac{1}{4}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{\pi}{6} + \frac{1}{2\sqrt{3}}$$

4

$$= \frac{\pi}{6} + \frac{\sqrt{3}}{6}$$

Note: Rationalise the denominator.

$$= \frac{1}{6}(\pi + \sqrt{3})$$

(b) $y = 3\sin^{-1}(2x)$

(i) Domain: $-1 \le 2x \le 1$

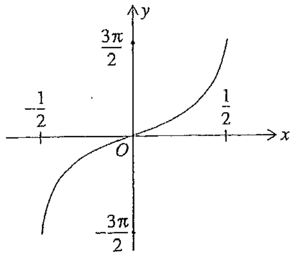
i.e. $-\frac{1}{2} \le x \le \frac{1}{2}$

Range: $-\frac{\pi}{2} \le \frac{y}{3} \le \frac{\pi}{2}$

i.e. $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$ **2**

Note: For $y = \sin^{-1}x$, domain is $-1 \le x \le 1$, range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

(ii)



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(iii) $\frac{dy}{dx} = 3 \times \frac{1}{\sqrt{1-4x^2}} \times 2$

$= \frac{6}{\sqrt{1-4x^2}}$ **1**

(iv) $\frac{dy}{dx}$ is defined for $-\frac{1}{2} < x < \frac{1}{2}$ as the tangents are vertical at $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ and so the derivative does not exist at the endpoints of the domain. **1**

Note: If this is not recognised then it can be shown by solving the inequality $1 - 4x^2 > 0$ giving $-\frac{1}{2} < x < \frac{1}{2}$.

QUESTION 3.

(a) $\int_0^{\frac{3}{2}} \frac{1}{\sqrt{9-2t^2}}\,dt = \int_0^{\frac{3}{2}} \frac{1}{\sqrt{2\left(\frac{9}{2}-t^2\right)}}\,dt$

$= \frac{1}{\sqrt{2}}\int_0^{\frac{3}{2}} \frac{1}{\sqrt{\left(\frac{3}{\sqrt{2}}\right)^2 - t^2}}\,dt$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\left(\frac{t}{\frac{3}{\sqrt{2}}}\right)\right]_0^{\frac{3}{2}}$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{t\sqrt{2}}{3}\right]_0^{\frac{3}{2}}$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}0\right]$

$= \frac{1}{\sqrt{2}} \times \left(\frac{\pi}{4} - 0\right)$

$= \frac{\pi}{4\sqrt{2}}$ **3**

Note: The formula in the HSC table of standard integrals is:

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}$.

Here $a^2 = \frac{9}{2}$, $a = \frac{3}{\sqrt{2}}$.

(b) $y = \frac{1}{\sqrt{9+x^2}}$, from $x = -3$ to $x = 3$

$V = \pi\int_a^b y^2\,dx$

$V = \pi\int_{-3}^{3}\left(\frac{1}{\sqrt{9+x^2}}\right)^2 dx$

$V = 2\pi\int_0^3 \frac{1}{9+x^2}\,dx$

$V = 2\pi\left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3$

$V = \frac{2\pi}{3}\left[\tan^{-1}\frac{x}{3}\right]_0^3$

$V = \frac{2\pi}{3}[\tan^{-1}(1) - \tan^{-1}(0)]$

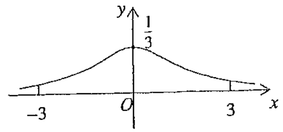
$V = \frac{2\pi}{3}\left[\frac{\pi}{4} - 0\right]$

$V = \frac{\pi^2}{6} \doteqdot 1.64$

$\therefore$ required volume is $\frac{\pi^2}{6}$ units^3.

(or 1.64 units^3 correct to two decimal places). **4**

Note: It is often important to draw a diagram.



Note: This is an even function (i.e. $f(x) = f(-x)$) which means the area on both sides of the y axis is the same. We can change the limits as shown.

Note: The formula in the HSC table of standard integrals is:

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$

Here $a^2 = 9$, $a = 3$

(c) To prove $\tan^{-1}\left(\frac{2}{3}\right)+\tan^{-1}\left(\frac{1}{5}\right)=\frac{\pi}{4}$.

Let $A=\tan^{-1}\frac{2}{3}$ and $B=\tan^{-1}\frac{1}{5}$.

$\therefore\ \tan A=\frac{2}{3}$ and $\tan B=\frac{1}{5}$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

Note: Multiply numerator and denominator by 15.

$$=\frac{\frac{2}{3}+\frac{1}{5}}{1-\frac{2}{3}\times\frac{1}{5}}$$

$$=\frac{10+3}{15-2}$$

$$=1$$

$$A+B=\frac{\pi}{4}$$

Note: Since A and B are acute.

$$\therefore\ \tan^{-1}\left(\frac{2}{3}\right)+\tan^{-1}\left(\frac{1}{5}\right)=\frac{\pi}{4}$$

3

QUESTION 4.

(a) $y=\cos^{-1}(2x-1)$

(i) Domain: $-1\le 2x-1\le 1$

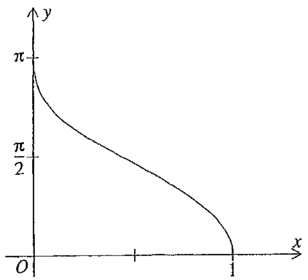
$0\le 2x\le 2$

$0\le x\le 1$

Range: $0\le y\le\pi$ **1**

Note: For $f(x)=\cos^{-1}x$, domain is $-1\le x\le 1$, range is $0\le\cos^{-1}x\le\pi$.

(ii)



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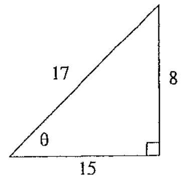
(b) Let $\theta=\cos^{-1}\frac{15}{17}$.

$\therefore\cos\theta=\frac{15}{17}$

Using a right-angle triangle and Pythagoras' theorem,

Note: θ is acute.



← by Pythagoras

$\therefore\ \sin\theta=\frac{8}{17}$

$$\sin 2\theta=2\sin\theta\cos\theta$$

$$=2\times\frac{8}{17}\times\frac{15}{17}$$

$$=\frac{240}{289}$$

$$\therefore\ \sin\left(2\cos^{-1}\frac{15}{17}\right)=\frac{240}{289}$$

3

(c) $y=-\cos^{-1}x$ .... (1)

$y=2\tan^{-1}(x-1)$ ... (2)

(i) Substitute $x=0$ into (1): $y=-\cos^{-1}0$

$y=-\frac{\pi}{2}$

Note: All points on the y-axis have an x-coordinate of 0.

Substitute $x=0$ into (2): $y=2\tan^{-1}(-1)$

$y=2\left(-\frac{\pi}{4}\right)$

$y=-\frac{\pi}{2}$

Hence the curves intersect at the same point $\left(0,-\frac{\pi}{2}\right)$ on the y axis. **2**

(ii) For $y = -\cos^{-1}x$

$$\frac{dy}{dx} = -\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

When $x = 0$, $\frac{dy}{dx} = \frac{1}{\sqrt{1-0^2}}$

$$= 1$$

For $y = 2\tan^{-1}(x-1)$

$$\frac{dy}{dx} = 2\frac{1}{(x-1)^2+1} \times 1$$

$$\frac{dy}{dx} = \frac{2}{(x-1)^2+1}$$

When $x = 0$, $\frac{dy}{dx} = \frac{2}{(0-1)^2+1} = 1$

$\therefore$ the tangents at $\left(0, -\frac{\pi}{2}\right)$ have the same gradient.

Hence the curves have a common tangent at this point. 2