SYDNEY GIRLS HIGH SCHOOL

LABOR OMNIA VINCIT

YEAR 12

Extension course 1

June 2002 (Assessment Three)

MATHEMATICS

Time allowed: 90 minutes

Topics Tested: Inverse Trigonometry, Integration by Substitution, Circle Geometry, Polynomials

Instructions:

- There are SIX (6) questions. Each question is of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work.
- Start each question on a new page. Write on one side of the paper only.

Name: ....

SGHS June 2002 Extension 1 paper:

Question 1 (10 marks)

a) Find $\int x\sqrt{x^2-1}\,dx$ using $u = x^2 - 1$

b) Find the remainder when $x^3 - 5x^2 + 2x - 3$ is divided by $x-2$

c) If $x^3 - 2x + 5 = 0$ has roots α, β and γ

Find i) $\alpha+\beta+\gamma$

ii) $\alpha\beta+\alpha\gamma+\beta\gamma$

iii) $\alpha.\beta.\gamma$

iv) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

v) $\alpha^2+\beta^2+\gamma^2$

Question 2 (10 marks)

a) Consider $P(x) = 2x^3 - 6x + 1$

i) Show $P(x)$ has a zero α between 1 and 2

ii) After examining $P'(1)$ and $P'(2)$ choose either 1 or 2 as the first approximation to the zero α, *justifying your choice.*

iii) Find a better approximation to the zero α, using Newton's method once with *your choice* of 1 or 2 as the first approximation.

b) There is a root of $f(x) = x^3 - 1 + \sin^{-1}x$ between 0 and 1. Use the halving the interval method once, determine which of 0 or 1 is the better approximation.

c) Two circles, radii 9cm and 25cm, touch externally.

i) Draw a neat diagram showing this information.

ii) Find the length of the common tangent.

Question 3 (10 marks)

a) Consider $F(x) = x^2 \sin^{-1} x$

i) What is the domain of $F(x)$?
ii) Find $F(x)$ for both endpoints and the midpoint of the domain.
iii) Show $F(x)$ is an odd function.
iv) Find $F'(x)$ and show $F'(0) = 0$
v) What happens to $F'(x)$ at the endpoints of the domain
vi) Hence sketch $F(x)$

b) Factor the polynomial $Q(x) = 15x^3 - 22x^2 + 5x + 2$ completely, and hence sketch the graph of $y = Q(x)$ showing where it cuts the x and y axes. (You do not need to use calculus.)

c) Find

i) $\displaystyle\int_{-2}^{2} \frac{dx}{\sqrt{4-x^2}}$ ii) $\displaystyle\int \frac{9.dx}{9+16x^2}$

Question 4 (10 marks)

a) Three circles, centres A, B and C, touch externally.
AB = 8cm, BC = 5 cm, CA = 7 cm.

i) Draw a neat diagram showing all information.

ii) Find the radius of circle A.

b) Sketch the graph of $f(x) = 6 \sin^{-1} \frac{x}{2}$

i) What is the domain of $f(x)$?
ii) What is the range of $f(x)$?
iii) The inverse of $f(x)$ is $f^{-1}(x)$. Is $f^{-1}(x)$ also a function?
iv) On a new set of axes, graph $f^{-1}(x)$ showing its range and domain.

Question 5 (10 marks)

a) Using $x = \sin u$ find $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$

b) Using $u = \log_e 2x$ show the area enclosed by $y = \dfrac{\log_e 2x}{x}$, the x-axis, and the lines $x = 2$ and $x = 4$ is equal to $\frac{5}{2}(\log_e 2)^2$

c) Using $u = x^2 + 6$, find $\int f(x)\,dx$ where $f(x) = x(x^2 + 6)^8$

Question 6 (10 marks)

a) Given $f(x) = 3x^2 + 5$

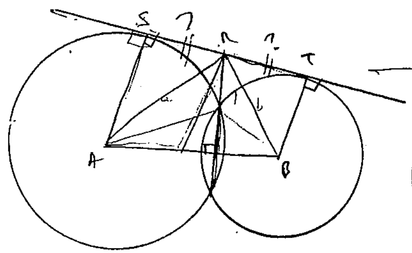
i) Write down the domain and range of $f(x)$
ii) Write down the equation of the inverse of $f(x)$
iii) Explain why the inverse is not a function.
iv) Suggest a restriction on the domain of $f(x)$ so that the inverse becomes a function.

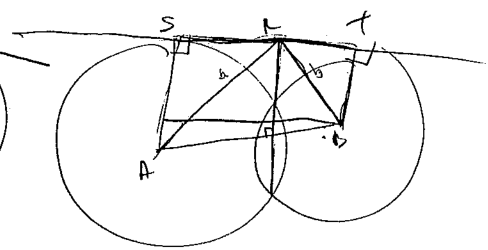
b) Two circles with centres A and B and different radii intersect at P and Q.
S lies on circle centre A and T lies on circle centre B.
ST is tangent to both circles. PQ produced meets ST at R.

i) Draw a neat diagram showing the information

ii) Prove that SR = RT

c) Find the derivative of $x\cos^{-1}x$ and hence evaluate $\displaystyle\int_0^1 \cos^{-1}x.dx$





3U June 2002

a) $\int x\sqrt{x^2-1}\,.dx$ $\quad u = x^2-1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$= \frac{1}{2}\int u^{1/2}\,du$

$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + c$

$= \frac{1}{3}(x^2-1)^{3/2} + c$ (3)

b) $f(x) = x^3-5x^2+2x-3$

$f(2) = 8 - 20 + 4 - 3$

$= -11$ (1)

c) $x^3-2x+5=0$ (6)

i) $\alpha+\beta+\gamma = 0$

ii) $\alpha\beta+\alpha\gamma+\beta\gamma = c/a = -2$

iii) $\alpha\beta\gamma = -d/a = -5$

iv) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\Sigma\alpha\beta}{\alpha\beta\gamma} = 2/5$

v) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2\,\Sigma\alpha\beta$

$= 4$

2a) $P(x) = 2x^3-6x+1$

i) $P(1) = 2-6+1 = -3$ (1)

$P(2) = 16-12+1 = 5$

Since $P(x)$ changes sign, there is a root between $x=1$ & $x=2$

ii) $P'(x) = 6x^2-6$

$P'(1) = 6-6 = 0$

$P'(2) = 24-6 = 18$

$\therefore$ take $P'(2)$ as $x=1$ is a stationary point.

$X_2 = X_1 - \frac{f(X_1)}{f'(X_1)}$

$\therefore X_2 = 2 - \frac{f(2)}{f'(2)}$

$= 2 - \frac{5}{18}$

$= 1\frac{13}{18}$ (4)

b) $f(x) = x^3-1+\sin^{-1}x$

$f(0) = 0-1+0 = -1$

$f(1) = 1-1+\sin^{-1}(1) = \pi/2$

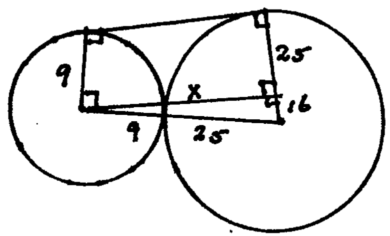
$f(\frac{1}{2}) = \frac{1}{8} - 1 + \sin^{-1}(\frac{1}{2}) = -\frac{7}{8}+\frac{\pi}{6}$

$\doteqdot -.351$

$\therefore$ Root lies between $x=\frac{1}{2}$ & $x=1$,

$\therefore$ root lies closer to $x=1$ (than $x=0$). (2)

c) (3)



$x^2 = 34^2 - 16^2$

$= (34+16)(34-16)$

$= 50\times 18$

$= 900$

$\therefore x = 30.$

3a) $F(x) = x^2\sin^{-1}x$

i) $-1 \le x \le 1$

ii) $F(-1) = \sin^{-1}(-1) = -\pi/2$

$F(0) = 0$

$F(1) = \sin^{-1}(1) = \pi/2$

iii) $F(-x) = (-x)^2\sin^{-1}(-x)$

$= -x^2\sin^{-1}(x)$

$= -F(x)$ $\therefore$ odd

iv) $F'(x) = 2x.\sin^{-1}x + x^2\frac{1}{\sqrt{1-x^2}}$

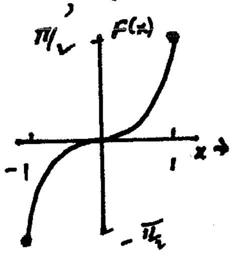
$F'(0) = 0.\sin^{-1}(0)+0$

$= 0$

v) as $x\to 1$, $F'(1) \to 2\sin^{-1}(1) + \frac{1}{\sqrt{0}}$

$\Rightarrow$ undefined

as $x \to -1$, $F'(-1)$ is also undefined



(4)

b) $Q(x) = 15x^3-22x^2+5x+2$

$Q(1) = 15-22+5+2 = 0$ $\therefore (x-1)$ is a factor

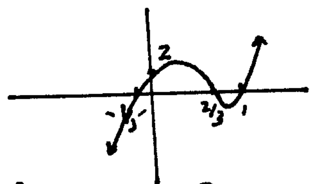
$$\begin{array}{r} 15x^2-7x-2 \\ x-1\,\overline{)\,15x^3-22x^2+5x+2} \\ 15x^3-15x^2 \\ \hline -7x^2+5x \\ -7x^2+7x \\ \hline -2x+2 \end{array}$$

-30

$\therefore Q(x) = (x-1)(15x^2-7x-2)$

$= \frac{(x-1)(15x-10)(15x+3)}{15}$

$= (x-1)(3x-2)(5x+1)$



(3)

c i) $\int_{-2}^{2}\frac{dx}{\sqrt{4-x^2}} = 2\left[\sin^{-1}(\frac{x}{2})\right]_0^2$

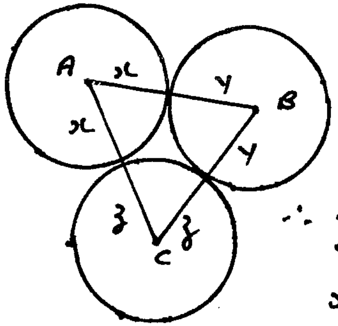
$= 2.\sin^{-1}(1)$

$= \pi$

ii) $\int\frac{9dx}{9+16x^2}$ (3)

$= \frac{\cancel{9}\,3}{3.4}\tan^{-1}(\frac{4x}{3}) + c$

4a)



$AB = 8$
$BC = 5$
$CA = 7$

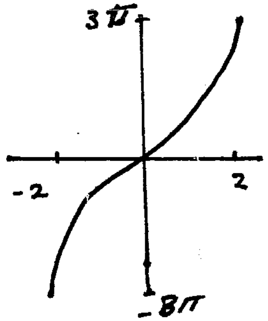
$\therefore x+y = 8$
$y+z = 5$
$x+z = 7$

$x - y = 3$

$\therefore 2x = 10$
$x = 5$

(4)

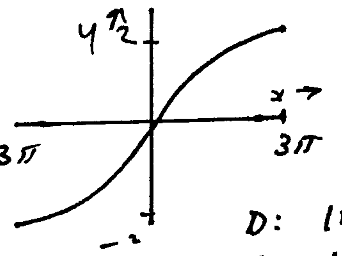
b)



D: $-2 \leq x \leq 2$
R: $-3\pi \leq y \leq 3\pi$

(6)

iii) $f^{-1}(x)$ is also an inverse since $f(x)$ has no stat. pts.



D: $|x| \leq 3\pi$
R: $|y| \leq 2$

5a) $\int_0^1 \sqrt{1-x^2}\,dx$ $\quad x = \sin u$, $dx = \cos u\,du$ $\quad \Big|\ x=0, u=0;\ x=1, u=\pi/2$

$$I = \int_0^{\pi/2} \sqrt{1-\sin^2 u}\,.\cos u\,du$$
$$= \int_0^{\pi/2} \cos^2 u\,du$$
$$= \frac{1}{2}\int_0^{\pi/2} (1+\cos 2u)\,du$$
$$= \frac{1}{2}\left[u + \frac{1}{2}\sin 2u\right]_0^{\pi/2}$$
$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{1}{2}.0\right] = \frac{1}{2}(\pi+0) = \frac{\pi}{4}$$

(4)

b) $\int_2^4 \frac{\ln 2x}{x}\,dx$ $\quad u = \ln 2x$, $du = \frac{dx}{x}$ $\quad \Big|\ x=2, u=\ln 4;\ x=4, u=\ln 8$

$$= \int_{\ln 4}^{\ln 8} u\,du$$
$$= \frac{1}{2}\left[u^2\right]_{\ln 4}^{\ln 8}$$
$$= \frac{1}{2}\left[(\ln 8)^2 - (\ln 4)^2\right]$$
$$= \frac{1}{2}\left[\ln 8 + \ln 4\right]\left[\ln 8 - \ln 4\right]$$
$$= \frac{1}{2}\ \ln 32 .\ \ln 2$$
$$= \frac{1}{2}.5\ln 2.\ln 2 = \frac{5}{2}(\ln 2)^2$$

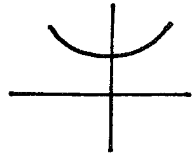
(4)

c) $\int x(x^2+6)^8\,dx$ $\quad u = x^2+6$, $du = 2x\,dx$, $\frac{1}{2}du = x\,dx$

$$I = \frac{1}{2}\int u^8\,du$$
$$= \frac{1}{2}\frac{u^9}{9} + c$$
$$= \frac{1}{18}(x^2+6)^9 + c$$

(2)

6a) $f(x) = 3x^2+5$



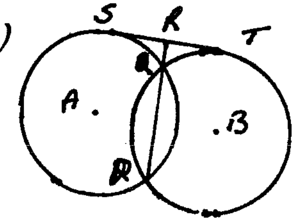
i) D: all real x
R: $y \geq 5$

ii) $x = 3y^2+5$
$\therefore \frac{x-5}{3} = y^2 \Rightarrow y = \pm\sqrt{\frac{x-5}{3}}$

iii) Not a function since there are 2 y values for each x value > 5.

iv) Restrict $f(x)$ to $x \geq 0$

b)



$SR^2 = RQ.RP$ (intersecting tangent & segments of chords)

similarly
$RT^2 = RQ.RP$

$\therefore SR^2 = RT^2$
$\therefore SR = RT$

(2)

c) $\frac{d}{dx}(x\cos^{-1}x) = 1.\cos^{-1}x - \frac{x}{\sqrt{1-x^2}}$

$\therefore \int \cos^{-1}x\,dx - \int \frac{x\,dx}{\sqrt{1-x^2}} = x\cos^{-1}x$

$\therefore \int \cos^{-1}x\,dx = x\cos^{-1}x + \int \frac{x\,dx}{\sqrt{1-x^2}}$

let $u = 1-x^2$, $\therefore du = -2x\,dx$, $-\frac{1}{2}du = x\,dx$

$\therefore \int \cos^{-1}x\,dx = x\cos^{-1}x - \frac{1}{2}\int u^{-\frac{1}{2}}du$
$= x\cos^{-1}x - u^{1/2}$
$= x\cos^{-1}x - \sqrt{1-x^2} + c$

$\therefore \int_0^1 \cos^{-1}x\,dx = \left[x\cos^{-1}x - \sqrt{1-x^2}\right]_0^1$
$= (\cos^{-1}1 - \sqrt{0}) - (0 - \sqrt{1})$
$= 1$