St Catherine's School

Year: 12
Subject: Extension I Mathematics
Time Allowed: 55 minutes

Date: June 2006

Exam number: 16361275

Directions to candidates:

- All questions are to be attempted.
- All necessary **working** must be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Approved calculators and geometrical instruments are required.
- Hand in your work in 1 **bundle**:
- Attach the question paper

Q.1. (i) Find $\int \frac{x\,dx}{\sqrt{x-5}}$, use the substitution $x-5=u$ (3m)

(ii) Find $\int_0^3 \sqrt{9-x^2}\,dx$, using the substitution $x=3\sin\theta$ (5m)

(iii) $\int \frac{x+1}{(x^2+2x-1)^2}dx$, using the substitution $x^2+2x-1=u$ (3m)

Q.2. A spherical balloon is being deflated so that the radius is decreasing at a constant rate of 8 mm per second. Find the rate of change of volume when the radius is 5 mm.
(Note that the volume of a sphere is given by $V=\frac{4\pi r^3}{3}$) (3m)

Q.3. The acceleration of a particle moving in Simple Harmonic motion is given by $\frac{d^2x}{dt^2}=-16x$. The particle starts at the centre of motion with a velocity of 3m/sec

(i) Using only the expression of acceleration and the initial conditions, show that $v^2=9-16x^2$ (2m)

(ii) Hence or otherwise, find an expression for x in terms of t, where x is the displacement from the origin at time t seconds. (3m)

(iii) Find the value of the acceleration at the end points of the motion. (2m)

P.T.O.

Q.4 . The cooling rate of a body is proportional to the difference between the temperature of the body and that of the surrounding medium and is represented by the equation $\frac{dT}{dt} = -k(T - M)$, where T is the temperature of the body and M is the surrounding temperature.

The original temperature of a body is 90^0 C and the temperature of the surrounding is 25^0C .It cools to 80^0C in 20 minutes.

(i) Show that $T = 25 + Ae^{-kt}$ is a solution to the given equation. (1m)

(ii) Show that $A = 65$ and $k = 0.0084$ (2 sig figs) (3m)

(ii) Find the temperature after 30 minutes. (1m)

(iii) Find the time taken to cool to 30^0C. (1m)

(iii) Sketch the graph of T **and** find the limiting value of the temperature. (2m)

Q.5 Sue hits a golf ball with a velocity of 50 metres per second and at an angle of α to the horizontal.

(i) Place the coordinate exes at the point of projection and show that the parametric expressions for x and y, the horizontal and vertical displacements respectively, in terms of t, are given by $x = 50 \cos\alpha\ t$ *and* $y = -5t^2 + 50 \sin\alpha\ t$

(Take g, the acceleration due to gravity as -10m/sec^2) (2m)

(ii) Find the value of α, the angle of projection, so that the ball just clears a wall 8 metres in height and 10 metres away. (4m)

END OF PAPER

ST CATHERINE'S SCHOOL JUNE '06

Q1 (i) $\int \frac{x}{\sqrt{x-5}}\,dx$

Let $x-5=u$

$dx=du$ ✓

$\int \frac{x}{\sqrt{x-5}}\,dx = \int \frac{5+u}{\sqrt{u}}\,du$

$= \int (5+u)(u^{-\frac{1}{2}})\,du$

$= \int 5u^{-\frac{1}{2}} + u^{\frac{1}{2}}\,du$ ✓

$= \left[\frac{2\times 5u^{\frac{1}{2}}}{1} + \frac{2u^{\frac{3}{2}}}{3}\right]$

$= 10\sqrt{u} + 2\sqrt{u^3}$

$= 10\sqrt{x-5} + 2\sqrt{(x-5)^3}$ ✓

$= 2\left(5\sqrt{x-5} + \sqrt{(x-5)^3}\right)$

(ii) $\int_0^3 \sqrt{9-x^2}\,dx$

Let $x=3\sin\theta \Rightarrow \theta = \sin^{-1}\frac{x}{3}$

$dx = 3\cos\theta\,d\theta$ ✓

When $x=0$, $\theta=0$

$x=3$, $\theta=\frac{\pi}{2}$

$\int_0^3 \sqrt{9-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta}\ .\ 3\cos\theta\,d\theta$

$= 3\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$

$= \frac{3}{2}\int_0^{\frac{\pi}{2}} 1+\cos 2\theta\,d\theta$ ✓

$= \frac{3}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$

$= \frac{3}{2}\left[\frac{\pi}{2}\right]$ ✓ $= \frac{3\pi^2}{4}$ ✓

(iii) $\int \frac{x+1}{(x^2+2x-1)^2}\,dx$

Let $u = x^2+2x-1$

$\frac{du}{dx} = 2x+2 = 2(x+1)$ ✓

$\int \frac{x+1}{(x^2+2x-1)^2}\,dx = \frac{1}{2}\int \frac{du}{u^2}$

$= \frac{1}{2}\int u^{-2}\,du$ ✓

$= \frac{1}{2}\left[\frac{u^{-1}}{-1}\right] + c$

$= \frac{-1}{2u} + c$

$= \frac{-1}{2(x^2+2x-1)} + c$ ✓

Q2 $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$\frac{dr}{dt} = 8\text{mm/s}$

$\frac{dV}{dr} = 4\pi r^2$

When $r=5$, $\frac{dV}{dr} = 4\pi \times 25 = 100\pi$ ✓

$\frac{dV}{dt} = 100\pi \times 8$

$= 800\pi\ \text{mm}^3/\text{s}$ ✓

Q3

(i) $\ddot{x} = -16x$

~~$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$~~

$\frac{1}{2}v^2 = \int -16x \; dx$

$= -8x^2 + c$ ✓

when $x = 0, v = 3$

$\frac{1}{2} \times 9 = 0 + c$

$c = \frac{9}{2}$ ✓

$\therefore \frac{1}{2}v^2 = -8x^2 + \frac{9}{2}$

$v^2 = -16x^2 + 9$ ✓

(ii) $v = \frac{dx}{dt} = \sqrt{9-16x^2}$

$\frac{dt}{dx} = \frac{1}{\sqrt{9-16x^2}}$

$t = \int \frac{1}{\sqrt{9-16x^2}} dx$

$= \frac{1}{\sqrt{16}} \int \frac{1}{\sqrt{\frac{9}{16}-x^2}} dx \quad = \frac{1}{4}\sin^{-1}\left(\frac{4x}{3}\right) + c$

~~$= \frac{1}{\sqrt{16}} \cdot \frac{4}{3} \tan^{-1}\frac{4x}{3} + c$~~ At $x=0, t=0, c=0$

~~$= \frac{1}{12}\tan^{-1}\frac{4x}{3} + c$~~ $\therefore t = \frac{1}{4}\sin^{-1}\left(\frac{4x}{3}\right)$

~~when $t=0, x=0$~~ $\boxed{x = \frac{3}{4}\sin 4t}$

~~$\therefore c = 0$~~

~~$12t = \tan^{-1}\frac{4x}{3}$~~

~~$\frac{4x}{3} = \tan 12t$~~ $\Rightarrow x = \frac{3}{4}\tan 12t$

(iii) At end pt of motion

$v = 0$

~~$9 - 16x^2 = 0$~~

$16\left(\frac{9}{16} - x^2\right) = 0$

$x^2 = \frac{9}{16}$ ✓

$x = \pm\frac{3}{4}$

$a = |-16x|$

$= \left|-16\left(\frac{3}{4}\right)\right|$

$= |\pm 12 \text{ ms}^{-2}| \quad = 12 \text{ m/s}^2$

Q4 $\frac{dT}{dt} = -k(T-M)$

$t = 0, T = 90$

$M = 25$

$t = 20, T = 80$

(i) $T = 25 + Ae^{-kt}$

$\frac{dT}{dt} = -k \cdot Ae^{-kt}$ ✓

$= -k(T-25)$ ✓

$\therefore T = 25 + Ae^{-kt}$ is a sol'n.

(i) $t=0$; $90 = 25 + Ae^{0}$

$A = 90 - 25$ ✓

$= 65$ ✓

$t = 20$; ~~[illegible]~~

$80 = 25 + 65e^{-20k}$

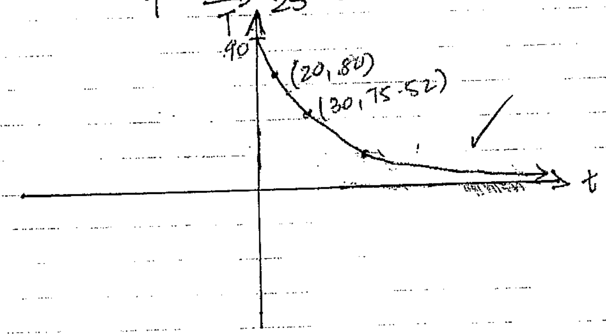
$\frac{55}{65} = e^{-20k}$ ✓

~~20~~ $k = \frac{-1}{20}\ln\left(\frac{55}{65}\right) \approx 0.0084$ (2sf) ✓

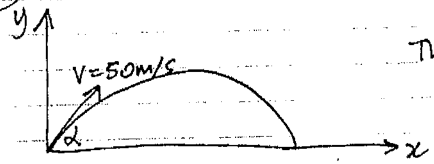
(iii) $t = 30$, $T = 25 + 65e^{-0.0084 \times 30}$
$= 75.52°$ (2dp)

(iv) $T = 30$, $30 = 25 + 65e^{-0.0084t}$
$\frac{5}{65} = e^{-0.0084t}$
$t = \frac{-1}{0.0084} \ln\left(\frac{1}{13}\right)$
$= 305.35$ mins

(v) as $t \to \infty$
$e^{-0.0084t} \to 0$
$\therefore T \to 25 + 0$
$T \to 25$



Q5 $V = 50$



The only acceleration is gravity.

$\ddot{x} = 0$
$\dot{x} = c$
$t=0, \dot{x} = V\cos\alpha$
$= 50\cos\alpha$
$\therefore \dot{x} = 50\cos\alpha$
$x = 50t\cos\alpha + c$
$t=0, x=0$
$\therefore x = 50t\cos\alpha$

$\ddot{y} = -10$
$\dot{y} = -10t + c$
$t=0, \dot{y} = 50\sin\alpha$
$\therefore c = 50\sin\alpha$
$\dot{y} = -10t + 50\sin\alpha$
$y = -5t^2 + 50t\sin\alpha + c$
$t=0, y=0 \quad \therefore c=0$
$\therefore y = -5t^2 + 50t\sin\alpha$

ii) when $x = 10$, $y = 8$

$10 = 50t\cos\alpha \implies t = \frac{10}{50\cos\alpha} = \frac{1}{5\cos\alpha}$

Sub in
$8 = -5t^2 + 50(\sin\alpha)t$
$8 = -5\left(\frac{1}{5\cos\alpha}\right)^2 + 50\left(\frac{1}{5\cos\alpha}\right)\sin\alpha$
$= -\frac{1}{5\cos^2\alpha} + \frac{10}{\cos\alpha} \cdot \sin\alpha$
$= -\frac{1}{5}\sec^2\alpha + 10\tan\alpha$
$= -\frac{1}{5}(1 + \tan^2\alpha) + 10\tan\alpha$
$-40 = 1 + \tan^2\alpha - 50\tan\alpha$
$\tan^2\alpha - 50\tan\alpha + 41 = 0$

$\tan\alpha = \frac{50 \pm \sqrt{50^2 - 4(41)}}{2}$

$\alpha = 88°50'$ or $39°50'$