St Vincent's College

Potts Point

Extension 1 Mathematics
Assessment 3
Term 2, 2009

NAME: ______________________________

Time allowed: 50 minutes

Working must be shown for all questions.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

1) State the range and domain of the function $y = 2\ sin^{-1}(\frac{x}{2})$ and draw a sketch of the function, carefully labelling the extremities of both the range and the domain. [3]

2) Evaluate $\int_0^{1\cdot 5} \frac{dx}{\sqrt{9-2x^2}}$, leaving your answer in exact form. [3]

3) Evaluate $\int_0^{\frac{\pi}{6}} \frac{2\cos x}{1+4\sin^2 x}\,dx$ using the substitution $u = 2\sin x$. [3]

4) i. If $f(x) = e^{x+2}$, find the inverse function $f^{-1}(x)$. [2]

ii. State the domain and range of $f^{-1}(x)$. [2]

iii. On one diagram sketch the graphs of $f(x)$ and $f^{-1}(x)$. [3]

5) A projectile is fired from a position 2m above ground level at an angle of elevation of 30 . The initial velocity is 1000m/s. [Use g = -10m/s/s]

i. Find the horizontal and vertical motion equations (velocity, acceleration, displacement) [3]

ii. Find the maximum height the projectile reaches [2]

i. Find the time taken to hit the ground. (Assume the ground is level for as long as needed) [2]

iv. Find the distance travelled by the projectile before it hits the ground. [Neglect air resistance and the curvature of the Earth] [1]

6) A particle is performing Simple Harmonic Motion in a straight line. At time t seconds it has velocity v metres per second, and displacement x metres from a fixed point O on the line, where $x = 5\cos\frac{\pi t}{2}$.

i. Find the period of the motion. [1]

ii. Find an expression for v in terms of t, and hence show that $v^2 = \frac{\pi^2}{4}(25 - x^2)$. [3]

iii. Find the speed of the particle when it is 4 metres to the right of O. [1]

7) O is a fixed point on a given straight line. A particle moves along this line and its displacement x cm, from O at a given time, t secs, after its start of motion is given by: $x = 2 + \cos^2 t$.

i. Show that the acceleration is given by: $\ddot{x} = 10 - 4x$. $2(5-2x)$ [2]

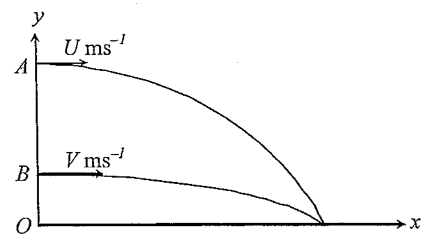
ii. State the centre of motion. [1]

iii. State the first two occasions when the particle is at rest and the displacements on these occasions [3]

8)

$OA = 4h$

$OB = h$ metres



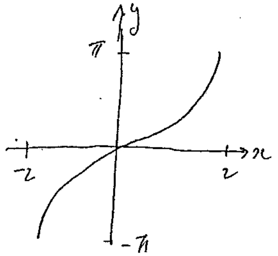
A vertical building stands with its base O on horizontal ground. A and B are two points on the building vertically above each other such that A is $4h$ metres above O and B is h metres above O. A particle is projected horizontally with speed U ms^{-1} from A and 10 seconds later a second particle is projected horizontally with speed V ms^{-1} from B. The two particles hit the ground at the same point and at the same time.

i. Write down expressions for the horizontal and the vertical displacements relative to O of each particle t seconds after the first particle is projected. [3]

ii. Find the time of flight of each particle. [3]

iii. Show that $V = 2U$. [1]

Q1



Domain $-2 \le x \le 2$

Range $-\pi \le y \le \pi$

Q2

$$\int_0^{1.5} \frac{dx}{\sqrt{9-2x^2}}$$

$$= \left[\frac{1}{\sqrt2}\sin^{-1}\frac{\sqrt2 x}{3}\right]_0^{1.5}$$

$$= \frac{1}{\sqrt2}\sin^{-1}\frac{1}{\sqrt2} - 0$$

$$= \frac{1}{\sqrt2}\cdot\frac{\pi}{4}$$

$$= \frac{\pi}{4\sqrt2}$$

Q 3)

$$\int_0^{\frac{\pi}{6}} \frac{2\cos x}{1+4\sin^2 x}\,dx$$

$$\boxed{\begin{array}{l} u = \sin x \\ du = \cos x\,dx \\ dx = \frac{du}{\cos x} \\ x=0,\ u=0 \\ x=\frac{\pi}{6}\ \ u=\frac12 \end{array}}$$

$$= \int_0^{\frac12} \frac{2}{1+4u^2}\,du$$

$$= \left[\tan^{-1} 2x\right]_0^{\frac12}$$

$$= \frac{\pi}{4}$$

Q4

(i) Let $y = e^{x+2}$

Interchanging x and y

$x = e^{y+2}$

$\ln x = y+2$

$y = (\ln x) - 2$

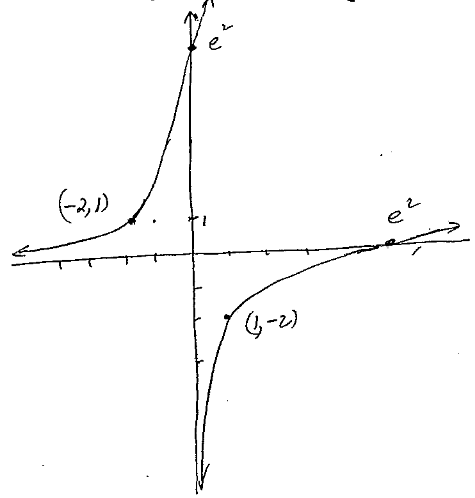
$\therefore f^{-1}(x) = (\ln x) - 2.$

(ii) $\ln x$ is defined for $x > 0$

$\therefore$ domain $f^{-1}(x)$ is $x > 0$

Range $f^{-1}(x)$ is all reals.

(iii)



(6) $x = 5\cos\frac{\pi t}{2}$

(i) $T = \frac{2\pi}{\pi/2}$

$= 4$

(ii)

$v = \frac{dx}{dt} = -\frac{5\pi}{2}\sin\frac{\pi t}{2}$

$v^2 = \frac{25\pi^2}{4}\sin^2\frac{\pi t}{2}$

$= \frac{25\pi^2}{4}\left(1-\cos^2\frac{\pi t}{2}\right)$

$= \frac{\pi^2}{4}\left(25 - 25\cos^2\frac{\pi t}{2}\right)$

$= \frac{\pi^2}{4}\left(25 - x^2\right)$

(iii) $v^2 = \frac{\pi^2}{4}(25-16)$

$= \frac{9\pi^2}{4}$

$v = \frac{3\pi}{2}.$

(7)

(i) $x = 2 + \cos^2 t$

$\dot{x} = -2\cos t\sin t$

$\dot{x} = -\sin 2t$

$\ddot{x} = -2\cos 2t$

$= -2(2\cos^2 t - 1)$

$= -2(2(x-2)-1)$

$\ddot{x} = -4x + 10$

7 (ii) When $\ddot{x} = 0$

$4x = 10$

$x = 2\frac12$

$\therefore$ Centre of motion is $2\frac12$.

OR $\ddot{x} = -4(x - 2\frac12)$

$\therefore$ Centre at $2\frac12$

(iii) at rest when $\dot{x} = 0$

$\sin 2t = 0$

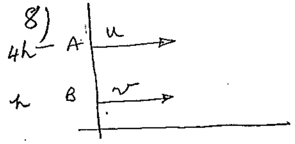
$t = 0,\quad t = \frac{\pi}{2}$

and displacement

$t=0,\quad x=3$

$t=\pi,\quad x=2.$

8)



(i)

$x_A = ut$

$\ddot{y}_A = -g$

$\dot{y} = -gt^2 + c$

But $c = 0$ $(\sin 0 = 0)$

$y_A = -5t^2 + 4h$

$x_B = v(t-10)$

$y_B = -5(t-10)^2 + h$

(ii) Time to hit ground

A) when $y = 0$

$5t^2 = 4h$

$t^2 = \frac{4h}{5}$

$t = \sqrt{\frac{4h}{5}}$

$= 2\sqrt{\frac{h}{5}}$

B) $5(t-10)^2 = h$

$(t-10)^2 = \frac{h}{5}$

$t - 10 = \sqrt{\frac{h}{5}}$

$t = 10 + \sqrt{\frac{h}{5}}$

times are same:–

Hence $2\sqrt{\frac{h}{5}} = 10 + \sqrt{\frac{h}{5}}$

$\therefore \sqrt{\frac{h}{5}} = 10$

$\frac{h}{5} = 100 \rightarrow h = 500$

If $h = 500$

$h = 500$

Then time for A is: $y = -5t^2 + 4h$

$5t^2 = 4(500)$

$t^2 = 400$

$t = 20$

and t for B:

$5(t-10)^2 = 500$

$(t-10)^2 = 100$

$t - 10 = 10$

$\underline{\underline{t = 20}}$

But starts 10 sec later $\therefore$ time for B = 10

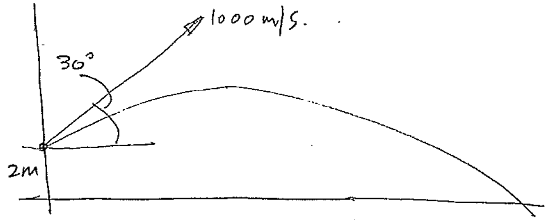
Speed $= \frac{D}{T}$

$\therefore u = \frac{D}{20}$ —①

and $v = \frac{D}{10}$

Hence $v = 2u$

Q 5



(i) $\dot{x} = 1000 \cos 30$

$= 500\sqrt{3}$

$\underline{x = 500t\sqrt{3}}$

$\ddot{y} = -10$

$\dot{y} = -10t + 500$

$y = -5t^2 + 500t + 2$

(ii) Max Height when $\dot{y} = 0$

ie. $10t = 500$

$t = 50$

$\therefore y(50) = -5(2500) + 500(50) + 2$

$= \underline{12502 \text{ m}}$

iii) Time to hit ground.

when $y = 0$

$-5t^2 + 500t + 2 = 0$

$t = \frac{-500 \pm \sqrt{250000 + 40}}{-10}$

$= \frac{-500 \pm 500.04}{10}$

$= 100.004$ sec ignore -ve answer

$\therefore$ (iv) Distance travelled is

$x = 500(100)\sqrt{3}$

$= \underline{\underline{86606 \text{ m}}}$