THE BINOMIAL THEOREM

1) Find (a) the number of terms and (b) the sum of coefficients of the binomial product $(a + b)^8$ without expanding.

2) Expand $(x - 2)^4$

3) Expand $\left(\frac{x}{3}+2y\right)^5$

4) Evaluate ${}^7C_4$

5) Evaluate $\binom{11}{5}$

6) Evaluate $\binom{8}{2}$

7) Find the coefficient of x^4 in the expansion of $\left(x+\frac{2}{x^2}\right)^{10}$

8) Find the constant term of $\sum_{k=0}^{7} {}^7C_k(3)^{7-k}(2x)^k$

9) Find the constant term of $\sum_{r=0}^{8}\binom{8}{r}x^{8-r}\left(\frac{3}{x}\right)^r$

10) Simplify ${}^9C_3+{}^9C_4$

11) Show that $\binom{n}{k}=\binom{n}{n-k}$

12) Simplify ${}^8C_5 \div {}^8C_4$

13) Write $(x - 3y)^{25}$ in sigma notation.

14) Find the 6th term of the expansion $\left(\frac{1}{x}+x^2\right)^9$

15) Find the 4th term of $(2 - \tan\theta)^7$

16) Find the term of $\left(\frac{3}{y^3}+\frac{y}{4}\right)^{12}$ that is independent of y.

17) Find values of a and b if the 5th term of $(ax + b)^{10}$ is $3360x^6$ and the 8th term is $15360x^3$ $(a > 0, b > 0)$.

18) C_k is the coefficient of the kth term of the binomial product $(3x + 2)^{15}$.

(a) Show that $\frac{C_{k+1}}{C_k}=\frac{2(16-k)}{3k}$

(b) Hence find the greatest coefficient of the expansion Leave your answer in the form $\binom{n}{k}a^{n-k}b^k$.

19) By using the fact that $(1+x)^4(1+x)^7=(1+x)^{11}$, show that

$$\binom{4}{0}\binom{7}{4}+\binom{4}{1}\binom{7}{3}+\binom{4}{2}\binom{7}{2}+\binom{4}{3}\binom{7}{1}+\binom{4}{4}\binom{7}{0}=\binom{11}{4}$$

20) Consider the binomial expansion

$(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\ldots+\binom{n}{n}x^n$. Show that

(a) $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\ldots+\binom{n}{n}=2^n$

(b) $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\ldots+(-1)^n\binom{n}{n}=0$

(c) $\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\ldots+n\binom{n}{n}=n2^{n-1}$

(d) $\binom{n}{0}-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\frac{1}{4}\binom{n}{3}+\ldots+(-1)^{n+1}\frac{1}{n+1}\binom{n}{n}=\frac{1}{n+1}$

21) Given $(1+x)^n(1+x)^n=(1+x)^{2n}$, show that

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\binom{n}{3}^2+\ldots+\binom{n}{n}^2=\binom{2n}{n}$$

ANSWERS

1) (a) 9 (b) 256

2) $x^4 - 8x^3 + 24x^2 - 32x + 16$

3) $\frac{x^5}{243} + \frac{10x^4y}{81} + \frac{40x^3y^2}{27} + \frac{80x^2y^3}{9} + \frac{80xy^4}{3} + 32y^5$

4) 35

5) 462

6) 28

7) 180

8) 2187

9) 5670

10) $^{10}C_4$

11) $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

$$\binom{n}{n-k} = \frac{n!}{[n-(n-k)]!(n-k)}$$

$$= \frac{n!}{k!(n-k)!}$$

$$= \binom{n}{k}$$

12) $\frac{4}{5}$

13) $\sum_{k=0}^{25}\binom{25}{k}x^{25-k}(-3y)^k$

14) $126x^6$

15) $-560\tan^3\theta$

16) $\frac{1485}{65536}$

17) a = 1, b = 2

18) (a) $\frac{\binom{15}{k}3^{15-k}2^k}{\binom{15}{k-1}3^{15-(k-1)}2^{k-1}}$

$$= \frac{\frac{15!}{(15-k)!k!}}{\frac{15!}{[15-(k-1)]!(k-1)!}} \times \frac{2}{3}$$

$$= \frac{15!}{(15-k)!k!} \times \frac{(16-k)!(k-1)!}{15!} \times \frac{2}{3}$$

$$= \frac{2(16-k)}{3k}$$

(b) $\binom{15}{6} 3^9 2^6$

19) $(1+x)^4(1+x)^7 = ({}^4C_0 + {}^4C_1x + {}^4C_2x^2 + {}^4C_3x^3 + {}^4C_4x^4)({}^7C_0 + {}^7C_1x + {}^7C_2x^2 + {}^7C_3x^3 + {}^7C_4x^4 + \ldots)$

The coefficient of x^4 is ${}^4C_0\,{}^7C_4 + {}^4C_1\,{}^7C_3 + {}^4C_2\,{}^7C_2 + {}^4C_3\,{}^7C_1 + {}^4C_4\,{}^7C_0$

In $(1 + x)^{11}$ the coefficient of x^4 is ${}^{11}C_4$

So ${}^4C_0\,{}^7C_4 + {}^4C_1\,{}^7C_3 + {}^4C_2\,{}^7C_2 + {}^4C_3\,{}^7C_1 + {}^4C_4\,{}^7C_0 = {}^{11}C_4$

20) (a) Let $x = 1$.

$$(1+1)^n = \binom{n}{0} + \binom{n}{1}1 + \binom{n}{2}1^2 + \binom{n}{3}1^3 + \ldots + \binom{n}{n}1^n$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n}$$

(b) Let $x = -1$.

$$(1-1)^n = \binom{n}{0} + \binom{n}{1}(-1) + \binom{n}{2}(-1)^2 + \binom{n}{3}(-1)^3 + \ldots + \binom{n}{n}(-1)^n$$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^n\binom{n}{n}$$

(c) $\frac{d}{dx}[(1+x)^n] = \frac{d}{dx}\left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n\right]$

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + 4\binom{n}{4}x^3 + \ldots + n\binom{n}{n}x^{n-1}$$

Let $x = 1$:

$$n(1+1)^{n-1} = \binom{n}{1} + 2\binom{n}{2}1 + 3\binom{n}{3}1^2 + 4\binom{n}{4}1^3 + \ldots + n\binom{n}{n}1^{n-1}$$

$$n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + 4\binom{n}{4} + \ldots + n\binom{n}{n}$$

(d) $\int[(1+x)^n]\,dx = \int\left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n\right]dx$

$$\frac{(1+x)^{n+1}}{n+1} + k = \binom{n}{0}x + \binom{n}{1}\frac{x^2}{2} + \binom{n}{2}\frac{x^3}{3} + \binom{n}{3}\frac{x^4}{4} + \ldots + \binom{n}{n}\frac{x^{n+1}}{n+1} + c$$

$$\frac{(1+x)^{n+1}}{n+1} = \binom{n}{0}x + \binom{n}{1}\frac{x^2}{2} + \binom{n}{2}\frac{x^3}{3} + \binom{n}{3}\frac{x^4}{4} + \ldots + \binom{n}{n}\frac{x^{n+1}}{n+1} + C$$

Let $x = 0$:

$$\frac{(1+0)^{n+1}}{n+1} = \binom{n}{0}0 + \binom{n}{1}\frac{0^2}{2} + \binom{n}{2}\frac{0^3}{3} + \binom{n}{3}\frac{0^4}{4} + \ldots + \binom{n}{n}\frac{0^{n+1}}{n+1} + C$$

$$\frac{1}{n+1} = C$$

$$\therefore \frac{(1+x)^{n+1}}{n+1} = \binom{n}{0}x + \binom{n}{1}\frac{x^2}{2} + \binom{n}{2}\frac{x^3}{3} + \binom{n}{3}\frac{x^4}{4} + \ldots + \binom{n}{n}\frac{x^{n+1}}{n+1} + \frac{1}{n+1}$$

Let $x = -1$:

$$\frac{(1-1)^{n+1}}{n+1} = \binom{n}{0}(-1) + \binom{n}{1}\frac{(-1)^2}{2} + \binom{n}{2}\frac{(-1)^3}{3} + \binom{n}{3}\frac{(-1)^4}{4} + \ldots + \binom{n}{n}\frac{(-1)^{n+1}}{n+1} + \frac{1}{n+1}$$

$$0 = -\binom{n}{0} + \frac{1}{2}\binom{n}{1} - \frac{1}{3}\binom{n}{2} + \frac{1}{4}\binom{n}{3} - \ldots + \frac{1}{n+1}(-1)^{n+1}\binom{n}{n} + \frac{1}{n+1}$$

$$\binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \frac{1}{4}\binom{n}{3} + \ldots + \frac{1}{n+1}(-1)^{n+1}\binom{n}{n} = \frac{1}{n+1}$$

21) The coefficient of x^n in $(1+x)^n\,(1+x)^n$ is given by

$${}^nC_0\,{}^nC_n + {}^nC_1\,{}^nC_{n-1} + {}^nC_2\,{}^nC_{n-2} + {}^nC_3C_{n-3} + \ldots + {}^nC_n\,{}^nC_0$$

$$= {}^nC_0\,{}^nC_0 + {}^nC_1\,{}^nC_1 + {}^nC_2\,{}^nC_2 + \ldots + {}^nC_n\,{}^nC_n$$

$$= ({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \ldots + ({}^nC_n)^2$$

In $(1+x)^{2n}$ the coefficient of x^n is ${}^{2n}C_n$

So ${}^{2n}C_n = ({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \ldots + ({}^nC_n)^2$