THE BINOMIAL THEOREM – WORKSHEET

COURSE/LEVEL

NSW Secondary High School Year 12 HSC Extension Mathematics. Syllabus reference: 17.1 – 17.3

1. Expand $(2x^3 + y)^4$ using the binomial theorem.

2. (a) For the binomial expansion of $(a+b)^n$ write down the $(k + 1)$th term, T_{k+1}.

   (b) Hence find the coefficient of x^3 in the expansion of $\left(x^2 - \dfrac{2}{x}\right)^8$.

3. (a) Factorise $1 + x + x^2 + x^3$.

   (b) Hence, or otherwise, show that the coefficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^3$ is 12.

4. Show that the numerically greatest coefficient in the expansion of $(2+3x)^{10}$ is 2 449 440. Also show that the greatest term when $x = -2$ is ${}^{10}C_8 2^2 6^8$.

5. Using the expansion of $(1+x)^n$, show that $2^n = \sum_{k=0}^{n} \binom{n}{k}$.

6. Differentiate both sides of the identity

$$(1+x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k$$

   and show that

$$\sum_{k=0}^{2n} k \binom{2n}{k} = n\,4^n$$

7. By evaluating $\int_0^2 (1-x)^{2n+1}\,dx$ in two different ways, prove that the following identity holds for all odd values of m:

$$2 - \frac{2^2}{2}\,{}^mC_1 + \frac{2^3}{3}\,{}^mC_2 - \ldots + (-1)^r \frac{2^{r+1}}{r+1}\,{}^mC_r + \ldots + (-1)^{\overset{m+1}{\cancel{m}}} \frac{2^{m+1}}{m+1}\,{}^mC_m = 0$$

8. Using the expansion $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ prove the following:

   (i) $1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^n \binom{n}{n} = 0$

   (ii) $1 - \dfrac{1}{2}\binom{n}{1} + \dfrac{1}{3}\binom{n}{2} - \ldots + \dfrac{(-1)^n}{n+1}\binom{n}{n} = \dfrac{1}{n+1}$.

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Simplify coefficients

1. $(2x^3+y)^4$

$= 4c_0(2x^3)^4 + 4c_1(2x^3)^3 y + 4c_2(2x^3)^2 y^2 + 4c_3(2x^3) y^3 + 4c_4 y^4$.

2. a) $(a+b)^n$: $T_{k+1} = {}^nC_k\, a^{n-k} b^k$

b) $T_{r+1} = {}^{9}C_r\,(x^2)^{9-r}\left(-\frac{2}{x}\right)^r$

$= {}^{9}C_r(-2)^r(x^2)^{9-r}(+x)^{-r} = {}^{9}C_r(-2)^r(x)^{18-2r-r}$

For coeff x^3 :

$18-3r = 3$. ✓ ~~$16-3r=3$~~ ~~$13=3r$~~ ? ~~$18-2r=r$~~ ~~$18=r$~~

$15 = 3r$

$r = 5$.

$T_6 = {}^9C_5(-2)^5 = \boxed{-4032}$ ✓

3. a) $1+x+x^2+x^3$. quicker to $\Longrightarrow$ $1(1+x)+x^2(1+x)$

$P(-1)=0$. $\therefore$ $(x+1)$ is a factor $= (1+x^2)(1+x)$

$$\begin{array}{r l}
 & x^2 + 1 \\
x+1 \,\big) & x^3 + x^2 + x + 1 \\
 & -(x^3+x^2) \\
 & \quad 0 + x + 1 \\
 & \qquad 0
\end{array}$$

$\therefore = (x+1)(x^2+1)$ ✓

b) $(1+x+x^2+x^3)^3 = ((x+1)^3(x^2+1))^3$

$= (3c_0x^3 + 3c_1x^2 + 3c_2x^1 + 3c_3(1))(3c_0(x^2)^3 + 3c_1(x^2)^2 + 3c_2(x^2) + 3c_3)$

coeff. of x^4 : ${}^3C_3 \times {}^3C_2$

$(3c_1 \times 3c_2) + (\cancel{3c_0 \times 3c_2}) = \underline{12}$ ✓

4. $(2+3x)^{10}$.

greatest coeff:

$\frac{T_{r+1}}{T_r} > 1$ $\frac{C(T_{r+1})}{C(T_r)} = \frac{10c_r(2)^{10-r}(3x)^r}{10c_{r-1}(2)^{10-r+1}(3x)^{r-1}}$

$(2+3x)^{10}$

$T_{r+1} = 10c_r\,(2)^{10-r}\,(3x)^r$; $T_r = 10c_{r-1}\,(2)^{10-r+1}\,(3x)^{r-1}$.

for greatest coeff:

$$\frac{C(r+1)}{C(r)} = \frac{10c_r\,(2)^{10-r}\,(3)^r}{10c_{r-1}\,(2)^{10-r+1}\,(3)^{r-1}} > 1$$

$$= \frac{\cancel{10!}\ \cancel{2^{10-r}}\ (\cancel{3^r})^{3}}{\cancel{r!}_{r}\,(\cancel{10-r})!} \times \frac{\cancel{(r-1)!}\,(\cancel{10-r+1})!^{(10-r+1)}\ \cancel{2^{10-r+1}}\ 3^{r-1}}{\cancel{10!}\ \cancel{2^{10-r+1}}_{2}\ \cancel{3^{r-1}}} > 1.$$

$$= \frac{3\,(11-r)}{2r} > 1.$$

$33 - 3r > 2r$.

$33 > 5r$

on $r < 6.6 \Rightarrow r = 6$.

$T_7 = 10c_6\,(2)^4\,(3)^6 = \underline{2449440}$

~~$T_7 = 2449440\,x^6 = 2449440\,(2)^6 = ???$~~

$\underline{a=2}$; $\underline{b = 3\times 2}$.

$$\frac{T_{r+1}}{T_r} = \frac{nc_r}{nc_{r-1}}\left(\frac{b}{a}\right) = \frac{3\,(11-r)}{2r}\times\left(3\right) > 1 \Rightarrow 33 - 3r > \frac{2r}{3}$$

$99 - 9r > 2r$

$99 > 11r \Rightarrow r < 9$; $\underline{r = 8}$

$\boxed{T_9 = 10c_8\,2^2\,6^8}$

greatest term

5. $(1+x)^n = nc_0x^0 + nc_1x^1 + nc_2x^2 \dots + nc_nx^n$

Sub $x=1$;

$(2)^n = nc_0 + nc_1 + nc_2 \dots + nc_n$.

$= \sum_{k=0}^{n} nc_k$.

6) $(1+x)^{2n} = \sum_{k=0}^{2n} (2nc_k)\,x^k = 2nc_0x^0 + 2nc_1x^1 + 2nc_2x^2 + 2nc_2x^3 + \dots 2nc_{2n}x^{2n}$.

diff:

$2n(1+x)^{2n-1} = \cancel{2nc_0} + 2nc_1 + 2(2nc_2x) + \dots 2n\,(2nc_{2n}x^{2n-1})$ — ①

~~$\div 2;\ n(1+x)^{2n-1} = nc_1 + 2nc_2x + 3nc_3x^2 + \dots n(2nc_{2n}x^{2n-1})$~~.

show that:

$n(4)^n = 2nc_0 + 1\,2nc_1 + 2(2nc_2) + 3(2nc_3) \dots + 2n(2nc_{2n})$

Sub $x=1$ into ①;

$2n(2)^{2n-1} = 2nc_1 + 2(2nc_2) \dots + 2n(2nc_{2n})$.

$\frac{\cancel{2}n\,(4)^n}{\cancel{2}} = {}_{\wedge}^{0\times(2nc_0)}\,2nc_1 + 2nc_2 \dots + 2n\,(2nc_{2n})$

$n\,4^n = \sum_{k=0}^{2n} k\,(2nc_k)$

7) $\int_0^2 (1-x)^{2n+1}\,dx = \int_0^2 {}^{2n+1}C_0 (-x)^0 + {}^{2n+1}C_1 (-x)^1 + {}^{2n+1}C_2 (-x)^2 \dots + {}^{2n+1}C_{2n+1} (-x)^{2n+1}\,dx$

$= \int_0^2 {}^{2n+1}C_0\, 1 - {}^{2n+1}C_1 x + {}^{2n+1}C_2 x^2 \dots + {}^{2n+1}C_{2n+1} (-x)^{2n+1}\,dx$

$$(1-x)^{2n+1} = {}^{2n+1}C_0 x^0 - {}^{2n+1}C_1 x^1 + {}^{2n+1}C_2 x^2 \dots + {}^{2n+1}C_{2n+1} (-x)^{2n+1}$$

Integrating both sides w.r.t. x

$$\left[\frac{(1-x)^{2n+2}}{2n+2}\right]_0^2 = \left[{}^{2n+1}C_0 x - {}^{2n+1}C_1 \frac{x^2}{2} + {}^{2n+1}C_2 \frac{x^3}{3} \dots + {}^{2n+1}C_{2n+1} \frac{(-x)^{2n+2}}{2n+2}\right]_0^2$$

$$\frac{(-1)^{2n+2}}{2n+2} - \frac{1^{2n+2}}{2n+2} =$$

If $m = 2n+1$

$$\frac{+1}{2n+2} - \frac{1}{2n+2} = 0 = {}^{2n+1}C_0\, 2 - {}^{2n+1}C_1 \frac{2^2}{2} + {}^{2n+1}C_2 \frac{2^3}{3} - \dots + {}^{2n+1}C_{2n+1} \frac{(-2)^{m+1}}{m+1}$$

$$0 = 2 - \frac{2^2}{2}\,{}^mC_1 + \frac{2^3}{3}\,{}^mC_2 - \dots + \underbrace{{}^mC_m \frac{(-2)^{m+1}}{m+1}}_{{}^mC_m \frac{(+2)^{m+1}(-1)^{m+1}}{m+1}}$$

$$\boxed{2 - {}^mC_1 \frac{2^2}{2} + {}^mC_2 \frac{2^3}{2} \dots + (-1)^{m+1} \frac{2^{m+1}}{m+1}\,{}^mC_m = 0}$$

$$\frac{10!\,(2)^{10-r}(3x)^{r}}{r!\,(10-r)!} \times \frac{(r-1)!\,(10-r+1)!\,(11-r)}{10!\,2^{10-r+1}\,(3x)^{r-1}}$$

$10 - r - 10 + r - 1$

$r - r + 1$

$$= \frac{3x\,(11-r)}{r\,(2)} > 1$$

$11 - r > 2r$

$11 > 3r$

$r < 3.6\ldots$

$r = 3.$

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8) $(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 \ldots + {}^nC_n x^n$ (1)

i) Prove that:

$1 - {}^nC_1 + {}^nC_2 - {}^nC_3 \ldots + (-1)^n({}^nC_n) = 0.$

Sub $x = -1$ into (1)

$(1-1)^n = {}^nC_0(-1)^0 + {}^nC_1(-1) + {}^nC_2(-1)^2 \ldots + {}^nC_n(-1)^n$

$$\boxed{0 = 1 - {}^nC_1 + {}^nC_2 - {}^nC_3 \ldots + {}^nC_n(-1)^n}$$

ii) $1 - \frac{1}{2}({}^nC_1) + {}^nC_2\left(\frac{1}{3}\right) - \ldots \frac{(-1)^n}{(n+1)}\,{}^nC_n = \frac{1}{n+1}$

Integrate (1);

$$\frac{(1+x)^{n+1}}{n+1} = x + {}^nC_1\frac{x^2}{2} + {}^nC_2\frac{x^3}{3} - \ldots + {}^nC_n\frac{x^{n+1}}{n+1} + c.$$ (2)

$c \downarrow \frac{1}{n+1}$

Sub $x = 0$ into (2);

$c = \frac{1}{n+1}.$

Sub $x = -1$ into (2);

$$0 = -1 + {}^nC_1 \times \frac{1}{2} - {}^nC_2\frac{(1)}{3} - + {}^nC_n\frac{(-1)^{n+1}}{n+1} + \frac{1}{n+1}$$

$$\therefore \boxed{\frac{1}{n+1} = 1 - {}^nC_1\frac{1}{2} + {}^nC_2\frac{1}{3} \ldots + {}^nC_n\frac{(-1)^{n+1}}{n+1}}$$