YEAR 12 EXTENSION 1 TEST 31-8-05

Projectiles, Probability, Permutations and Combinations, Binomial Theorem.

Name___________________________Class_________

Instructions: Show all necessary working throughout the test on A4 paper.

Begin a new page as specified. **Time allowed: 50 minutes**

[Begin a new page] **HRK**

1.

(a) Find the value of the term independent of x in the expansion of

$$(2x - \frac{1}{x^2})^{12}$$ 3

(b) By comparing coefficients of x^4 in both sides of

$(1 + x)^4(1 + x)^4 = (1 + x)^8$, show that

$$\sum_{k=0}^{4} \left({}^4C_k\right)^2 = {}^8C_4$$ 3

(c) (i) Write down the Binomial expansion of $(1 + x)^n$ in ascending powers of x 1

Hence show that ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$ 2

(ii) Write down the expanded form of $\sum_{k=1}^{n-1} {}^nC_k$ 1

(iii) Show that ${}^nC_1 + {}^nC_2 + \ldots + {}^nC_{n-1} = 2^n - 2$ 1

(d) A machine is known to produce items of which 5% are too short and 95% are satisfactory. A random sample of twenty items is taken from the production of the machine.

Find the probability (correct to two decimal places) that:

(i) all of these items are satisfactory 1

(ii) at least eighteen of these items are satisfactory 3

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2. (a) The letters of the term "DELICIOUS FEAST" are arranged randomly in a row:

   (i) prove that the number of different arrangements is 10 897 286 400 — 2

   (ii) determine the number of ways that the vowels and consonants can alternate. — 2

(b) At a round table there are 3 boys and 7 girls.

   (i) In how many ways can the 10 people be seated at random? — 1

   (ii) If 3 of the girls wish to be seated next to one another, in how many ways can this seating arrangement be accommodated? — 2

   (iii) If a particular girl Anna is not to be seated between two particular boys Alexander and James, in how many ways can this seating arrangement be accommodated? — 3

(c) Consider a pack of 40 playing cards consisting of the colours Red, Blue, Yellow and Green, with cards numbered from 1 to 10 for each colour. If five cards are dealt at random from the pack find:

   (i) The total number of five card arrangements. — 1

   (ii) The probability of receiving three 4's and two 9's. — 2

   (iii) The probability of receiving five cards whose numbers are consecutive e.g. 3,4,5,6 and 7. — 2

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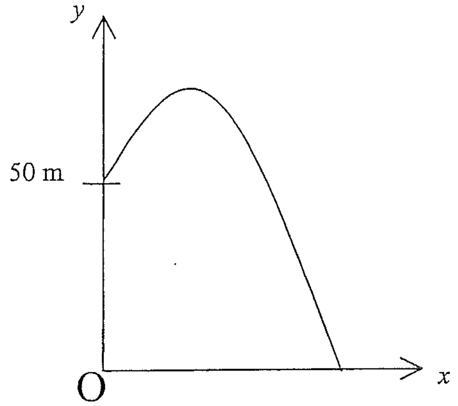
3. (a)



The diagram shows the path of a ball that is projected from the top of a tower 50 metres high. Its position t seconds after it is thrown is given by the equations:
$x = 20t$ and $y = 50 + 15t - 5t^2$ where the origin O is on the ground directly below the point of projection.

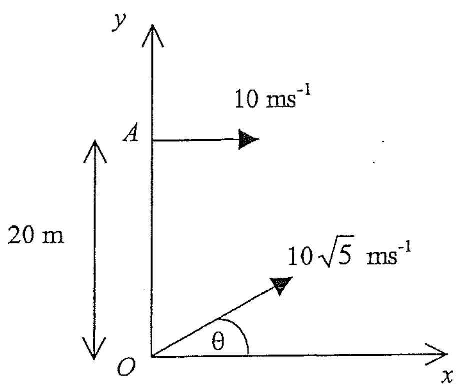
(i) Find the speed of projection **2**

(ii) Find the length of time before the ball strikes the ground. **1**

(iii) Calculate the maximum height above the ground reached by the ball. **2**

(iv) At what angle to the horizontal in the positive direction of the x-axis does the ball strike the ground? Give your answer to the nearest degree. **2**

(b)



OA is a vertical building of height 20 metres. A particle is projected <u>horizontally</u> from A with speed 10ms^{-1}. At the same instant another particle is projected from O with speed $10\sqrt{5}\ \text{ms}^{-1}$ at an angle θ above the horizontal. The two particles travel in the same plane of motion. Take $g = 10\ \text{ms}^{-1}$.

(i) <u>Derive</u> expressions for the horizontal and vertical displacements relative to O of each particle after t seconds. **4**

(ii) Show that if the two particles collide, then they do so after 1 second. **2**

(iii) Show that if the two particles collide, when they do so their paths of motion are perpendicular to each other. **2**

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1 (a) $T_r = {}^{12}C_{r-1}(2x)^{13-r}\left(\frac{-1}{x^2}\right)^{r-1}$

$= {}^{12}C_{r-1}\, 2^{13-r}\, x^{13-r}(-1)^{r-1}x^{-2r+2}$

$= {}^{12}C_{r-1}\, 2^{13-r}(-1)^{r-1}x^{15-3r}$

For term independent of x:

$15-3r=0 \quad \therefore r=5$

$\therefore T_5 = {}^{12}C_4\, 2^8(-1)^4$

$= 126720$

$\Rightarrow$ term independent of x is 126720

(b) RHS $=(1+x)^8$

$= {}^8C_0 + {}^8C_1x + \dots + {}^8C_4x^4 + \dots$
$+ {}^8C_8x^8$

LHS $=(1+x)^4(1+x)^4$

$= ({}^4C_0 + {}^4C_1x + {}^4C_2x^2 + {}^4C_3x^3 + {}^4C_4x^4)$
$\cdot({}^4C_0 + {}^4C_1x + {}^4C_2x^2 + {}^4C_3x^3 + {}^4C_4x^4)$

Comparing corr. coeffs of x^4 of LHS and RHS: ${}^8C_4 = {}^4C_0{}^4C_4 + {}^4C_1{}^4C_3$
$+ {}^4C_2{}^4C_2 + {}^4C_3{}^4C_1$
$+ {}^4C_4{}^4C_0$

$\therefore {}^8C_4 = ({}^4C_0)^2 + ({}^4C_1)^2 + ({}^4C_2)^2$
$+ ({}^4C_3)^2 + ({}^4C_4)^2$

$= \sum_{k=0}^{4}({}^4C_k)^2$

(c) (i) $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots$
$\dots + {}^nC_{n-1}x^{n-1} + {}^nC_nx^n$

let $x=1$ $\therefore (1+1)^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$

$\therefore 2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$

(ii) $\sum_{k=1}^{n-1}{}^nC_k = {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_{n-1}$

(iii) From (i) and (ii)

$2^n = {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_{n-1} + \underbrace{{}^nC_0 + {}^nC_n}$

Now as ${}^nC_0 = {}^nC_n = 1$

$\therefore 2^n - 2 = {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_{n-1}$

(d) Consider $(q+p)^{20}$

where $p = P(\text{satisfactory}) = 0.95$

$q = P(\overline{\text{satisfactory}}) = 0.05$

(i) $P(\text{all satisfactory}) = {}^{20}C_{20}\,p^{20}$

$= (0.95)^{20}$

$= 0.36$ (2dp)

(ii) P(at least 18 are satisfactory)

$= P(18 \text{ satis}) + P(19 \text{ satis}) + P(20 \text{ satis})$

$= {}^{20}C_{18}q^2p^{18} + {}^{20}C_{19}qp^{19} + {}^{20}C_{20}p^{20}$

$= {}^{20}C_{18}(0.05)^2(0.95)^{18} + {}^{20}C_{19}(0.05)(0.95)^{19}$
$+ {}^{20}C_{20}(0.95)^{20}$

$= 0.92$ (2dp)

2 (a) 'DELICIOUS FEAST'

has 14 letters consisting of the vowels 'E' ×2, 'I' ×2, 'A', 'O' and 'U' and the consonants 'D', 'L', 'C', 'S' ×2, 'F' and 'T'.

(i) No. of different arrangements

$$= \frac{14!}{2!.2!.2!}$$

$$= 10\,897\,286\,400$$

(ii) If vowels and consonants alternate ∴ no of arrangements

$$= \frac{2 \times 7! \times 7!}{2!.2!.2!}$$

$$= 6\,350\,400$$

(b) $\{3B, 7G\}$

(i) No. of ways $= 1 \times 9!$

$= 362\,880$

(ii) Consider the 3 girls as a unit.

This is achieved in 3! ways.

This leaves $(10-3)+1 = 8$ people units

∴ No. of ways $= 1 \times 7! \times 3!$

$= 30240$

(iii) No of ways without restrictions $= 9!$

No. of ways Anna between Alexander & James

$= 1 \times 2! \times 7!$

∴ No. of ways Anna not between Alex & James

$= 9! - 1 \times 2! \times 7!$

$= 352\,800$

(c) (i) No. of arrangements $= {}^{40}C_5$

$= 658\,008$

(ii) $P(3 \times 4\text{'s and } 2 \times 9\text{'s}) = \dfrac{{}^4C_3 \times {}^4C_2}{{}^{40}C_5}$

$$= \frac{24}{658008}$$

$$= \frac{1}{27417} \quad \left(\text{or} \doteqdot 3.65 \times 10^{-5}\right)$$

(iii) $P(\text{numbers are consecutive}) = \dfrac{(10-5+1) \times ({}^4C_1)^5}{{}^{40}C_5}$

$$= \frac{6144}{658008}$$

$$= \frac{384}{41125}$$

$(\text{OR} \doteqdot 9.34 \times 10^{-3})$

3 (a) $x = 20t$ —— ①

$y = 50 + 15t - 5t^2$ —— ②

(i) $v^2 = \dot{x}^2 + \dot{y}^2$

$= (20)^2 + (15 - 10t)^2$

At speed of projection $t = 0$

$\therefore v^2 = 20^2 + 15^2$

$= 625$

$\therefore v = 25$

$\therefore$ speed of projection is 25 ms^{-1}.

(ii) Ball strikes ground when $y = 0$

$\therefore 0 = 50 + 15t - 5t^2$

$\therefore 0 = 5t^2 - 15t - 50$

$0 = 5(t^2 - 3t - 10)$

$0 = 5(t-5)(t+2)$

$\therefore t = 5 \quad (t \geq 0)$

$\therefore$ Ball strikes ground after 5 secs.

(iii) At max. hgt $\dot{y} = 0$

$\therefore 15 - 10t = 0$

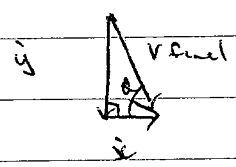
$\therefore t = 1\frac{1}{2}$

when $t = 1\frac{1}{2}$ $\quad y = 50 + 15(1\frac{1}{2}) - 5(1\frac{1}{2})^2$

$= 61\frac{1}{4}$

$\therefore$ Max. height reached by ball is $61\frac{1}{4}$ m.

(iv)



On ground: $\tan\theta = \left|\frac{\dot{y}}{\dot{x}}\right|$

$= \left|\frac{15 - 10t}{20}\right|$

At ground $t = 5$ $\therefore \tan\theta = \left|\frac{-35}{20}\right|$

$\therefore \angle\theta = 60°$ (to nearest deg)

$\therefore$ Ball strikes ground at approx. $120°$ to the horizontal in the positive direction of the x-axis.

(b)(i) For particle projected from A:

$\dot{x} = 10, \ \dot{y} = 0$

$\therefore \ddot{x} = 0, \ \ddot{y} = -10$

$\therefore \dot{x} = c_1, \quad \dot{y} = -10t + c_2$

when $t = 0$ $\quad \dot{x} = 10, \ \dot{y} = 0$

$\therefore c_1 = 10, \ c_2 = 0$

$\therefore \dot{x} = 10, \quad \dot{y} = -10t$

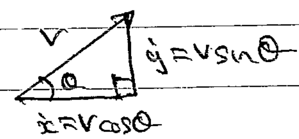
$\therefore x = 10t + c_3, \quad y = -5t^2 + c_4$

when $t = 0$ $\quad x = 10 \quad y = 20$

$\therefore 0 = c_3, \quad 20 = c_4$

$\therefore x = 10t + 10, \quad y = -5t^2 + 20$

For particle projected from O:



$\therefore \ddot{x} = 0, \quad \ddot{y} = -10$

$\therefore \dot{x} = c_1, \quad \dot{y} = -10t + c_2$

when $t = 0$ $\quad \dot{x} = V\cos\theta \quad \dot{y} = V\sin\theta$

$\therefore V\cos\theta = c_1, \quad V\sin\theta = c_2$

$\therefore \dot{x} = V\cos\theta, \quad \dot{y} = -10t + V\sin\theta$

$\therefore x = Vt\cos\theta + c_3, \quad y = -5t^2 + Vt\sin\theta + c_4$

when $t = 0$ $\quad x = 0, \ y = 0$

$\therefore c_3 = c_4 = 0$

$\therefore x = Vt\cos\theta, \quad y = -5t^2 + Vt\sin\theta$

Now as $V = 10\sqrt{5}$

$\therefore x = 10\sqrt{5}\,t\cos\theta, \quad y = -5t^2 + 10\sqrt{5}\,t\sin\theta$

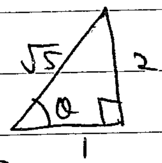
(ii) At point of collision of particles

$$10t = 10\sqrt{5}t\cos\theta \quad (1)$$

and $$-5t^2+20 = -5t^2+10\sqrt{5}t\sin\theta \quad (2)$$

Now from (1) $\cos\theta = \frac{1}{\sqrt{5}}$

$\therefore$ Triad is:



$\therefore \sin\theta = \frac{2}{\sqrt{5}}$ sub into (2)

$$\therefore\ 20 = 10\sqrt{5}t\cdot\frac{2}{\sqrt{5}}$$

$$\therefore\ t = 1$$

$\Rightarrow$ If particles collide they do so after 1 second.

(iii) Now $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$

$\therefore$ For particle at A: $\frac{dy}{dx} = -10t\cdot\frac{1}{10}$
$= -t$

At $t=1$ $\frac{dy}{dx} = -1 = m_{\text{tangent A}}$

For particle at O: $\frac{dy}{dx} = \frac{-10t+10\sqrt{5}\sin\theta}{10\sqrt{5}\cos\theta}$

At $t=1$ $\frac{dy}{dx} = \frac{-10+10\sqrt{5}\left(\frac{2}{\sqrt{5}}\right)}{10\sqrt{5}\left(\frac{1}{\sqrt{5}}\right)}$

$$= \frac{10}{10}$$

$$= 1 = m_{\text{tangent O}}$$

As $m_{\text{tangent A}}\cdot m_{\text{tangent O}} = -1$

$\Rightarrow$ paths of motion are perpendicular to each other at time of collision.