Yr 12 Ext 1 Revision Sheet 1: Inverse Functions and Integration by substitution

Inverse Functions

1(a) Given $f(x)=x^2-2x$:

(i) Sketch $f(x)$

(ii) State the domain and range of $f(x)$

(iii) Find the domain over which $f(x)$ is monotonic increasing

(iv) Find the equation of $y=f^{-1}(x)$ by using the restricted domain above,

(v) Sketch $y=f^{-1}(x)$

(vi) State the domain and range of $y=f^{-1}(x)$

(b) Find the inverse of the following functions:

(i) $f(x)=6\cos(2-7x)$ (ii) $f(x)=8\sin(6x-9)$ (iii) $f(x)=\tan^{-1}(e^x)$

2. Find the equation of the normal to the curve $y=\sin^{-1}(x-1)$ at the point where $x=\frac{3}{2}$

3. Sketch the following functions, stating their domain and range:

(a) $y=5\cos^{-1}3x$

(b) $y=2\sin^{-1}\left(\frac{x+2}{5}\right)$

(c) $y=4\tan^{-1}5x$

4. Evaluate $2\tan^{-1}(1)+2\tan^{-1}(-\sqrt{3})$, in terms of π.

5. Find the exact value of:

(a) $\sin^{-1}(-\frac{1}{\sqrt{2}})$ (b) $\cos^{-1}(\cos\frac{2\pi}{3})$ (c) $\tan\left[\sin^{-1}\left(-\frac{2}{3}\right)\right]$ (d) $\sin(2\tan^{-1}\frac{1}{2})$

6. Find the first derivative of the following:

(a) $y=(\sin^{-1}x)^2$ (b) $y=x^3\sin^{-1}\left(\frac{x}{2}\right)$ (c) $y=e^{\cos^{-1}(2x-1)}$

7. Find:

(a) $\int\frac{-dx}{\sqrt{2-x^2}}$ (b) $\int\frac{dx}{1+4x^2}$

8. Evaluate the following:

(a) $\int_0^{\sqrt{2}}\frac{dx}{2+x^2}$ (b) $\int_{\frac{1}{5\sqrt{2}}}^{\frac{\sqrt{3}}{10}}\frac{dx}{\sqrt{1-25x^2}}$

9.

(a) Find the exact area under the curve $y=\frac{1}{9+2x^2}$ bounded by the x-axis and the lines $x=\frac{\sqrt{3}}{\sqrt{2}}$ and $x=\frac{3}{\sqrt{2}}$

(b) The curve $y=\frac{3}{\sqrt{x^2+4}}$ is rotated about the x-axis between $x=\frac{2}{\sqrt{3}}$ and $x=2$. Find the volume of the solid generated.

10. Find the general solution for each of the following:

(a) $2\sin x=\sqrt{3}$ (b) $\sin 2x=\cos x$ (c) $\tan x=\sqrt{3}$ (d) $\tan 2\theta=\cot\theta$

Integration by substitution

11. Find:

(a) $\int\frac{3x}{\sqrt{2x^2-1}}dx$ where $u=2x^2-1$ (b) $\int x\sqrt{2-x}\ dx$ where $u=2-x$

12. Evaluate:

(a) $\int_4^9\frac{x}{\sqrt{x}-1}dx$, where $u=\sqrt{x}-1$ (b) $\int_{-5}^{3}x\sqrt{4-x}\,dx$, where $u=4-x$

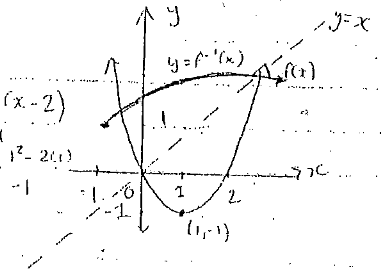
13. Find the area bounded by the curve $y=x^3(x^4-2)$, the x-axis and the lines $x=0$ and $x=1$.

14. Find the volume of the solid of revolution formed when the curve $y=x(x^3-3)^2$ is rotated about the x-axis from $x=0$ to $x=1$.

Yr 12 Ex1 Revision Sheet 1:
Inverse Functions & Integrat. by Substitution

1. (a) $f(x) = x^2 - 2x$

(i)



$f(x) = x(x-2)$
if $x = 1$
$f(1) = 1^2 - 2(1)$
$= -1$

(ii) D: all real x
R: $y \geq -1$

(iii) Domain: $x \geq 1$

(iv) Let $y = x^2 - 2x$
$x = y^2 - 2y$
$x + 1 = y^2 - 2y + 1$
$= (y-1)^2$
$\sqrt{x+1} = y - 1$
$\sqrt{x+1} + 1 = y$

(v) see above in part (i)

(vi) D: $x \geq -1$
R: $y \geq 1$

(b)(i) $f(x) = 6\cos(2-7x)$
Let $y = 6\cos(2-7x)$
$x = 6\cos(2-7y)$
$\frac{x}{6} = \cos(2-7y)$
$\cos^{-1}\left(\frac{x}{6}\right) = 2 - 7y$
$\cos^{-1}\left(\frac{x}{6}\right) - 2 = -7y$

$\therefore y = \dfrac{\cos^{-1}\left(\frac{x}{6}\right) - 2}{-7}$

$\therefore f^{-1}(x) = \dfrac{\cos^{-1}\left(\frac{x}{6}\right) - 2}{-7}$

(ii) $f(x) = 8\sin(6x-9)$
Let $y = 8\sin(6x-9)$
$x = 8\sin(6y-9)$
$\frac{x}{8} = \sin(6y-9)$
$\sin^{-1}\left(\frac{x}{8}\right) = 6y - 9$
$\sin^{-1}\left(\frac{x}{8}\right) + 9 = 6y$
$\dfrac{\sin^{-1}\left(\frac{x}{8}\right) + 9}{6} = y$

$\therefore f^{-1}(x) = \dfrac{\sin^{-1}\left(\frac{x}{8}\right) + 9}{6}$

(iii) $f(x) = \tan^{-1}(e^x)$
$y = \tan^{-1}(e^x)$
$x = \tan^{-1}(e^y)$
$\tan x = e^y$
$\ln(\tan x) = y$
$\therefore f^{-1}(x) = \ln(\tan x)$

2. $y = \sin^{-1}(x-1)$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x-1)^2}} \times 1$
$= \frac{1}{\sqrt{1-(x^2-2x+1)}}$
$= \frac{1}{\sqrt{-x^2+2x}}$

At $x = \frac{3}{2}$
$\frac{dy}{dx} = \frac{1}{\sqrt{-\left(\frac{3}{2}\right)^2 + 2\left(\frac{3}{2}\right)}}$
$= \frac{1}{\sqrt{\frac{3}{4}}}$
$= \frac{1}{\frac{\sqrt{3}}{2}}$

$m_1 = \frac{2}{\sqrt{3}}$

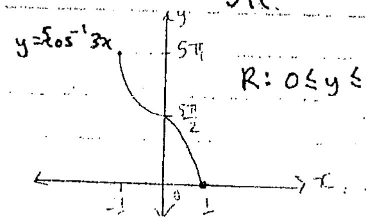
$m_2 = -\frac{\sqrt{3}}{2}$ if $x = \frac{3}{2}$, $y = \frac{\pi}{6}$

$y - y_1 = m_2(x - x_1)$
$y - \frac{\pi}{6} = -\frac{\sqrt{3}}{2}\left(x - \frac{3}{2}\right)$
$y = -\frac{\sqrt{3}}{2}\left(x - \frac{3}{2}\right) + \frac{\pi}{6}$

3. (a) D: $-1 \leq 3x \leq 1$
$-\frac{1}{3} \leq x \leq \frac{1}{3}$

if $x = \frac{1}{3}$ $y = 5\cos^{-1}3\left(\frac{1}{3}\right)$
$= 5\cos^{-1}1$
$= 0$

if $x = -\frac{1}{3}$ $y = 5\cos^{-1}3\left(-\frac{1}{3}\right)$
$= 5\pi$



R: $0 \leq y \leq 5\pi$

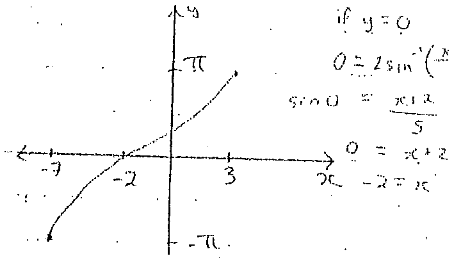
(b) $y = 2\sin^{-1}\left(\dfrac{x+2}{5}\right)$

D: $-1 \leq \frac{x+2}{5} \leq 1$
$-5 \leq x+2 \leq 5$
$-7 \leq x \leq 3$

if $x = 3$ $y = 2\sin^{-1}(1)$
$= \pi$

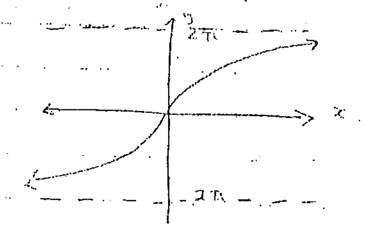
if $x = -7$ $y = 2\sin^{-1}(-1)$
$= -\pi$

$\therefore$ R: $-\pi \leq y \leq \pi$



if $y = 0$
$0 = 2\sin^{-1}\left(\frac{x}{}\right.$
$\sin 0 = \frac{x+2}{5}$
$0 = x+2$
$-2 = x$

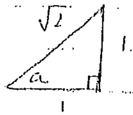
(c) $y = 4\tan^{-1}5x$
D: all real x
R: $-\frac{\pi}{2} < \frac{y}{4} < \frac{\pi}{2}$
$-2\pi < y < 2\pi$



4(a) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\pi$

Let $a=\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\sin a=\frac{1}{\sqrt{2}}$



$\cos a=\frac{1}{\sqrt{2}}$

$\therefore a=\frac{\pi}{4}$

$\therefore \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}+\frac{\pi}{4}$

$=\frac{\pi}{2}$

(b) $2\tan^{-1}(1)+2\tan^{-1}(\sqrt{3})$

$=2\left[\frac{\pi}{4}\right]+\left(-\tan^{-1}\sqrt{3}\right)$

$=\frac{\pi}{2}-\frac{2\pi}{3}$

$=-\frac{\pi}{6}$

5(a) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Let $y=\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

so $\sin y=-\frac{1}{\sqrt{2}}$

$\therefore y=-\frac{\pi}{4}$

(b) $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)=y$

Now $\cos(\cos^{-1}x)=x$ for $-1\le x\le 1$

so $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)=\frac{2\pi}{3}$

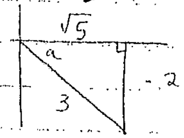
$\therefore y=\frac{2\pi}{3}$

(c) $\tan\left[\sin^{-1}\left(-\frac{2}{3}\right)\right]$

Let $a=\sin^{-1}\left(-\frac{2}{3}\right)$

$\sin a=-\frac{2}{3}$

Note: $-\frac{\pi}{2}\le a\le\frac{\pi}{2}$ so draw 'a' in 4th quad



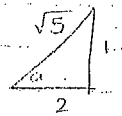
Now $\tan a=\frac{-2}{\sqrt{5}}$

$\therefore \tan\left[\sin^{-1}\left(-\frac{2}{3}\right)\right]=\frac{-2}{\sqrt{5}}$

(d) $\sin\left[2\tan^{-1}\left(\frac{1}{2}\right)\right]$

Let $a=\tan^{-1}\frac{1}{2}$

$\therefore \tan a=\frac{1}{2}$



$\therefore \sin 2a=2\sin a\cos a$

$=2\left(\frac{1}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}\right)$

$=\frac{4}{5}$

6(a) $y=(\sin^{-1}x)^2$

$\frac{dy}{dx}=2(\sin^{-1}x)\cdot\frac{1}{\sqrt{1-x^2}}$

$=\frac{2\sin^{-1}x}{\sqrt{1-x^2}}$

(b) $y=x^3\sin^{-1}\left(\frac{x}{2}\right)$

Let $u=x^3$ $\quad u'=3x^2$

$v=\sin^{-1}\left(\frac{x}{2}\right)$ $\quad v'=\frac{1}{\sqrt{4-x^2}}$

$\frac{dy}{dx}=\frac{x^3}{\sqrt{4-x^2}}+\sin^{-1}\left(\frac{x}{2}\right).3x^2$

(c) $y=e^{\cos^{-1}(2x-1)}$

$\frac{dy}{dx}=f'(x)\,e^{f(x)}$

$=\frac{d}{dx}\left[\cos^{-1}(2x-1)\right]e^{\cos^{-1}(2x-1)}$

$=\frac{-1}{\sqrt{1-(2x-1)^2}}\times 2\,.\,e^{\cos^{-1}(2x-1)}$

$=\frac{-2e^{\cos^{-1}(2x-1)}}{\sqrt{1-(4x^2-4x+1)}}$

$=\frac{-2e^{\cos^{-1}(2x-1)}}{\sqrt{4x-4x^2}}$

$=\frac{-2e^{\cos^{-1}(2x-1)}}{\sqrt{4x(1-x)}}$

$=\frac{-2e^{\cos^{-1}(2x-1)}}{2\sqrt{x(1-x)}}$

$=\frac{-e^{\cos^{-1}(2x-1)}}{\sqrt{x(1-x)}}$

7(a) $\int\frac{-1}{\sqrt{2-x^2}}dx$

$=\cos^{-1}\frac{x}{\sqrt{2}}+c$

(b) $\int\frac{dx}{1+4x^2}$

$=\int\frac{dx}{4\left(\frac{1}{4}+x^2\right)}$

$=\frac{1}{4}\int\frac{dx}{\left[x^2+\left(\frac{1}{2}\right)^2\right]}$

$=\frac{1}{4}\cdot\frac{1}{\frac{1}{2}}\left[\tan^{-1}2x\right]+c$

$=\frac{1}{2}\tan^{-1}2x+c$

8.(a) $\int_0^{\sqrt{2}}\frac{dx}{2+x^2}$

$=\frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{x}{\sqrt{2}}\right]_0^{\sqrt{2}}$

$=\frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{\sqrt{2}}{\sqrt{2}}-\tan^{-1}0\right]$

$=\frac{1}{\sqrt{2}}\left[\tan^{-1}1-\tan^{-1}0\right]$

$=\frac{1}{\sqrt{2}}\cdot\frac{\pi}{4}$

$=\frac{\pi}{4\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$

$=\frac{\pi\sqrt{2}}{8}$

(b) $\int_{\frac{1}{5\sqrt{2}}}^{\frac{\sqrt{3}}{10}}\frac{dx}{\sqrt{1-25x^2}}$

$=\int_{\frac{1}{5\sqrt{2}}}^{\frac{\sqrt{3}}{10}}\frac{dx}{\sqrt{25\left(\frac{1}{25}-x^2\right)}}$

$=\frac{1}{5}\int^{\frac{\sqrt{3}}{10}}\frac{dx}{\ldots}$

$= \frac{1}{5}\left[\sin^{-1} 5x\right]_{\frac{1}{5\sqrt{2}}}^{\frac{\sqrt{3}}{10}}$

$= \frac{1}{5}\left[\sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{\sqrt{2}}\right]$

$= \frac{1}{5}\left[\frac{\pi}{3} - \frac{\pi}{4}\right]$

$= \frac{1}{5}\cdot\frac{\pi}{12}$

$= \frac{\pi}{60}$

9(b) $V = \pi\int_{\frac{2}{\sqrt{3}}}^{2}\left(\frac{3}{\sqrt{x^2+4}}\right)^2 dx$

$= \pi\int_{\frac{2}{\sqrt{3}}}^{2}\frac{9}{x^2+4}dx$

$= 9\pi\cdot\frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_{\frac{2}{\sqrt{3}}}^{2}$

$= \frac{9\pi}{2}\left[\tan^{-1}1 - \tan^{-1}\frac{1}{\sqrt{3}}\right]$

$= \frac{9\pi}{2}\left[\frac{\pi}{4} - \frac{\pi}{6}\right]$

$= \frac{9\pi}{2}\cdot\frac{\pi}{12}$

$= \frac{3\pi^2}{8}u^3$

9(a) $A = \int_{\frac{\sqrt{3}}{\sqrt{2}}}^{\frac{3}{\sqrt{2}}}\frac{dx}{9+2x^2}$

$= \frac{1}{2}\int_{\frac{\sqrt{3}}{\sqrt{2}}}^{\frac{3}{\sqrt{2}}}\frac{dx}{\frac{9}{2}+x^2}$ Note: $a = \frac{3}{\sqrt{2}}$

$= \frac{\sqrt{2}}{3}\cdot\frac{1}{2}\left[\tan^{-1}\frac{\sqrt{2}x}{3}\right]_{\frac{\sqrt{3}}{\sqrt{2}}}^{\frac{3}{\sqrt{2}}}$

$= \frac{\sqrt{2}}{6}\left[\tan^{-1}1 - \tan^{-1}\frac{\sqrt{3}}{3}\right]$

$= \frac{\sqrt{2}}{6}\left[\frac{\pi}{4} - \frac{\pi}{6}\right]$

$= \frac{\sqrt{2}}{16}\cdot\frac{\pi}{12}$

$= \frac{\sqrt{2}\pi}{72}u^2$

10(a) $2\sin x = \sqrt{3}$

$\sin x = \frac{\sqrt{3}}{2}$

$\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$

Using the rule

$x = \pi n + (-1)^n\sin^{-1}b$

$\therefore x = \pi n + (-1)^n\frac{\pi}{3}$

(b) $\sin 2x = \cos x$

$2\sin x\cos x = \cos x$

$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ or $2\sin x = 1$

$\cos^{-1}0 = 1$ $\sin x = \frac{1}{2}$

Using the rule $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$\theta = 2\pi n \pm \cos^{-1}b$

$\therefore x = 2\pi n \pm \cos^{-1}0$ or $x = \pi n + (-1)^n\frac{\pi}{6}$

$= 2\pi n \pm \frac{\pi}{2}$

(c) $\tan x = \sqrt{3}$

$\tan^{-1}\sqrt{3} = \frac{\pi}{3}$

$\therefore x = \pi n + \frac{\pi}{3}$

(d) $\tan 2\theta = \cot\theta$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

so $\frac{2\tan\theta}{1-\tan^2\theta} = \frac{1}{\tan\theta}$

$2\tan^2\theta = 1 - \tan^2\theta$

$3\tan^2\theta = 1$

$\tan^2\theta = \frac{1}{3}$

$\therefore \tan\theta = \frac{1}{\sqrt{3}}$ or $\tan\theta = -\frac{1}{\sqrt{3}}$

$\therefore \theta = n\pi + \frac{\pi}{6}$ or $\theta = n\pi - \frac{\pi}{6}$

11(a) $\int\frac{3x\,dx}{\sqrt{2x^2-1}}$

using $u = 2x^2 - 1$

$\frac{du}{dx} = 4x$ $\therefore du = 4x\,dx$

$\therefore \int\frac{3x}{\sqrt{2x^2-1}}dx = \frac{3}{4}\int\frac{du}{\sqrt{u}}$

$= \frac{3}{4}\int u^{-\frac{1}{2}}du$

$= \frac{3}{4}\left[2u^{\frac{1}{2}}\right] + C$

$= \frac{3}{2}(2x^2-1)^{\frac{1}{2}} + C$

$= \frac{3}{2}\sqrt{2x^2-1} + C$

(b) $\int x\sqrt{2-x}\,dx$, $u = 2-x$

$u = 2-x$

$\frac{du}{dx} = -1$

$\therefore du = -1\,dx$

Now $u = 2-x$

$\therefore x = 2-u$

$\int x\sqrt{2-x}\,dx = -\int(2-u)u^{\frac{1}{2}}du$

$= -\int 2u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du$

$= -\left[2u^{\frac{3}{2}}\cdot\frac{2}{3} - \frac{2u^{\frac{5}{2}}}{5}\right] + C$

$= -\frac{4}{3}u^{\frac{3}{2}} + \frac{2u^{\frac{5}{2}}}{5} + C$

$= -\frac{4}{3}(2-x)^{\frac{3}{2}} + \frac{2}{5}(2-x)^{\frac{5}{2}} + C$

$= -\frac{4}{3}\sqrt{(2-x)^3} + \frac{2}{5}\sqrt{(2-x)^5} + C$

12.(a) $\int_4^9\frac{x}{\sqrt{x}-1}dx$

$u = \sqrt{x} - 1$ $\Rightarrow u+1 = \sqrt{x}$

$= x^{\frac{1}{2}} - 1$ $\boxed{(u+1)^2 = x}$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$

if $x = 4$, $u = 1$; if $x = 9$, $u = 2$

$\therefore du = \frac{dx}{2\sqrt{x}}$

$2\sqrt{x}\,du = dx$

$\boxed{2(u+1)\,du = dx}$ ... ②

$\therefore \int_1^2\frac{(u+1)^2\cdot 2(u+1)\,du}{(u+1)-1}$

$= 2\int_1^2\frac{(u+1)^3}{u}du$

$= 2\int_1^2\frac{u^3+3u^2+3u+1}{u}du$

$= 2\int_1^2 u^2 + 3u + 3 + \frac{1}{u}\,du$

$= 2\left[\frac{u^3}{3} + \frac{3u^2}{2} + 3u + \ln u\right]_1^2$

$= 2\left[\left(\frac{8}{3} + \frac{3}{2}(4) + 3(2) + \ln 2\right) - \left(\frac{1}{3} + \frac{3}{2} + 3 + c\right.\right.$

$= \frac{59}{3} + 2\ln 2$