Year 12 Extension 1 Mathematics

Term 3, 2007 Week 7

Name: ____________________

+ Solns

HSC ASSESSMENT TASK

50 Minutes Total Marks: 36

Teacher: ______________

ESSE QUAM VIDERI

Question 1 (12 Marks) **Marked by SKB**

Having been despatched to the boundary on the previous two balls, Angas, the crafty medium pace bowling cricketer, then bowls a slower ball at 15 m/s to the elegant right-hander Prinya. Prinya mishits this ball from a point O on the pitch at a velocity of 25m/s inclined at an angle θ to the positive direction of the pitch ($x-$axis). The ball just misses a bird flying at an altitude of 10m which is 5m horizontally from O. If we assume that air resistance is neglected and that the acceleration due to gravity, $g = 10ms^{-2}$:

a) Prove that the parametric equations describing the ball's motion are: $x = 25t\cos\theta$ and $y = 25t\sin\theta - 5t^2$ **2**

b) Prove that θ satisfies the equation $\tan^2\theta - 25\tan\theta + 51 = 0$ **2**

c) Find the value(s) of θ to the nearest minute. **2**

d) By using your answer(s) to (c) find the shortest time for the ball to reach the pitch in seconds to 2 decimal places. **2**

e) By using your answer(s) to (c) find the maximum height reached by the ball to the nearest m. **2**

f) Find the possible directions of motion when the ball hits the ground to the nearest minute. **2**

Question 2 (12 Marks) **Start a new page** **Marked by HRK**

a) Show that $\left(x^2 - \frac{1}{x}\right)^{11}$ does not contain a term independent of x. **3**

b) Find the greatest coefficient in the expansion of $(2+3x)^{10}$ **3**

c) Use the binomial expansion of $(1+x)^{2n}$ to prove that $\sum_{k=1}^{2n} \binom{2n}{k} = 4^n - 1$ **3**

d) An athlete finds that over a long period he wins $\frac{3}{4}$ of his races. If he intends running in 5 races over the coming weeks, what is the probability that he wins the majority of them? **3**

Question 3 (12 Marks) **Start a new page** **Marked by JSH**

a) How many arrangements of the word GOBBLEDEGOOK are possible? **2**

b) Consider a pack of 40 cards consisting of the colours red, blue, yellow and green, each with cards numbered 1 through to 10. A hand of five cards is dealt from the pack. Find the probabilities that the cards were

(i) three 4's and two 6's **1**

(ii) in ascending order **2**

c) Seven people are to be seated at a round table.

(i) How many arrangements are possible? **1**

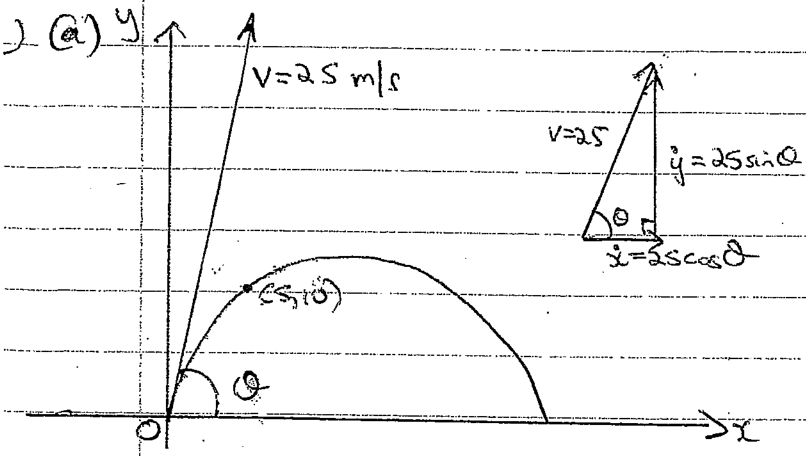
(ii) Two people, Kevin and Jill, refuse to sit next to each other. How many seating arrangements are possible? **3**

d) A committee of 3 is to be elected from a club of 8 members.

(i) How many different committees can be formed? **1**

(ii) If there are 4 Queenslanders in the club, what is the probability that a randomly selected committee of 3 contains only Queenslanders? **2**

1 (a)



Initially $\ddot{x}=0$, $\ddot{y}=-10$

$\therefore \dot{x}=c_1$, $\dot{y}=-10t+c_2$

when $t=0$ $\dot{x}=25\cos\theta$, $\dot{y}=25\sin\theta$

$\therefore 25\cos\theta=c_1$, $25\sin\theta=c_2$

$\therefore \dot{x}=25\cos\theta$, $\dot{y}=-10t+25\sin\theta$

$\therefore x=25t\cos\theta+c_3$, $y=-5t^2+25t\sin\theta+c_4$

when $t=0$, $x=0$, $y=0$ $\therefore c_3=0=c_4$

$\therefore$ parametric equations are:

$x=25t\cos\theta$ —— (1)

and $y=25t\sin\theta-5t^2$ —— (2)

(b) Ball just misses bird at (5,10)

$\therefore 5=25t\cos\theta$ —— (1A)

$10=25t\sin\theta-5t^2$ —— (2A)

from (1A) $t=\frac{1}{5\cos\theta}$ sub into (2A)

$\therefore 10=25\left(\frac{1}{5\cos\theta}\right)\sin\theta-5\left(\frac{1}{5\cos\theta}\right)^2$

$\therefore 10=5\tan\theta-\frac{\sec^2\theta}{5}$

$\therefore 50=25\tan\theta-(1+\tan^2\theta)$

$\therefore \tan^2\theta-25\tan\theta+51=0.$

(c) Now $\tan\theta=\frac{25\pm\sqrt{25^2-4.1.51}}{2}$

$=\frac{25\pm\sqrt{421}}{2}$

$\therefore \theta=87°29'$ or $65°57'$ (to nearest minute)

(d) Ball hits ground when $y=0$

$\therefore 0=25t\sin\theta-5t^2$

$\therefore 0=5t(5\sin\theta-t)$

$\therefore t=0$ (initial) or $t=5\sin\theta$ (final)

When $\theta \doteq 87°29'$ $t=5.00$ (2dp)

$\theta \doteq 65°57'$ $t \doteq 4.57$ (2d.p)

$\therefore$ shortest time for ball is 4.57s (2d.p)

(e) At maximum height $\dot{y}=0$

$\therefore t=\frac{25\sin\theta}{10}=\frac{5\sin\theta}{2}$

$\therefore y_{max}=25\left(\frac{5\sin\theta}{2}\right)\sin\theta-5\left(\frac{5\sin\theta}{2}\right)^2$

$=\frac{125\sin^2\theta}{2}-\frac{125\sin^2\theta}{4}$

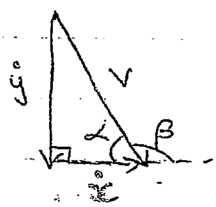
$=\frac{125\sin^2\theta}{4}$

Max height will occur for $\theta \doteq 87°29'$ as $\sin^2\theta$ will be greater.

$\therefore y_{max} \doteq \frac{125(\sin 87°29')^2}{4}$

$=31$ m (to nearest m)

(f) Ball hits ground after $5\sin\theta$ s from (iv).



$\therefore \dot{x}=25\cos\theta$

$\dot{y}=-10(5\sin\theta)+25\sin\theta$

$=-25\sin\theta$

Now $\tan\alpha=\left|\frac{\dot{y}}{\dot{x}}\right|=\tan\theta$ $\therefore \alpha=\theta$

$\therefore \beta$, direction of motion, is either $180°-87°29'$ or $180°-65°57'$

ie $92°31'$ or $114°3'$ (to nearest minute)

2(a) For $\left(x^2 - \frac{1}{x}\right)^{11}$

$$T_{r+1} = {}^{11}C_r \,(x^2)^{11-r}\left(-\frac{1}{x}\right)^r$$
$$= {}^{11}C_r \, x^{22-2r-r}(-1)^r$$
$$= {}^{11}C_r \, x^{22-3r}(-1)^r$$

For term independent of x: $22-3r=0$

$\therefore r = 7\frac{1}{3}$

But r must be a positive integer or zero

$\Rightarrow \left(x^2 - \frac{1}{x}\right)^{11}$ does not contain a term independent of x.

(b) For $(2+3x)^{10}$

consider $\dfrac{\text{coeff } T_{r+1}}{\text{coeff. } T_r} > 1$

for greatest coefficient.

$$\therefore \frac{{}^{10}C_r \, 2^{10-r}(3x)^r}{{}^{10}C_{r-1}\, 2^{11-r}(3x)^{r-1}} > 1$$

$$\therefore \frac{10!\,(11-r)!\,(r-1)!}{(10-r)!\,r!\,10!}\cdot\frac{3}{2} > 1$$

(ignoring x terms)

$$\therefore \frac{(11-r)3}{(r)2} > 1$$

$\therefore 33-3r > 2r$

$\therefore 33 > 5r \quad \therefore r < 6\frac{3}{5}$

$\therefore r = 6$

$\therefore$ greatest coefficient is: ${}^{10}C_6 \, 2^4 \, 3^6$

$= 2449440$

(c) $(1+x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k$

Let $x=1$ $\quad\therefore 2^{2n} = \sum_{k=0}^{2n}\binom{2n}{k}.1^k$

$$\therefore 4^n = \binom{2n}{0} + \sum_{k=1}^{2n}\binom{2n}{k}$$

$$\therefore 4^n - 1 = \sum_{k=1}^{2n}\binom{2n}{k}, \text{ as } \binom{2n}{0} = 1$$

(d) Consider $(q+p)^5$

where $P(\text{wins}) = p = \frac{3}{4}$

$P(\text{loses}) = q = \frac{1}{4}$

$\therefore P(\text{wins majority of races})$

$$= {}^5C_3\, q^2p^3 + {}^5C_4\, q p^4 + {}^5C_5\, p^5$$
$$= {}^5C_3\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)^3 + {}^5C_4 \cdot \tfrac{1}{4}\left(\tfrac{3}{4}\right)^4 + \left(\tfrac{3}{4}\right)^5$$
$$= 10\left(\frac{27}{1024}\right) + 5\left(\frac{81}{1024}\right) + \frac{243}{1024}$$
$$= \frac{459}{512} \quad (\text{or } 0.896484375\ldots)$$

3 (a) GOBBLEDEGOOK contains 12 letters including 2G's, 3O's, 2B's, 2E's, 1L, 1D, 1K.

$\therefore$ Number of arrangements $= \dfrac{12!}{2!\,3!\,2!\,2!}$

$= 9979200$

(b)(i) $P(3\ 4\text{'s}, 2\ 6\text{'s}) = \dfrac{{}^4C_3 \times {}^4C_2}{{}^{40}C_5}$

$= \dfrac{1}{27417}$

(ii) $P(\text{in ascending order}) = \left[(10-5+1) \times {}^4C_1 \times {}^4C_1 \times {}^4C_1 \times {}^4C_1 \times {}^4C_1\right] \div {}^{40}C_5$

$= \dfrac{6 \times 4^5}{{}^{40}C_5}$

$= \dfrac{256}{27417}$

3 (c) (i) No. of arrangements $= 1 \times 6!$
$= 720$

(ii) Consider Kevin & Jill to sit together. This is achieved in $2!$ ways. This leaves $(7-2)+1$
$= 6$ people units.

$\therefore$ No of arrangements Kevin & Jill together $= 1 \times 5! \times 2!$

$\therefore$ No of ways Kevin & Jill not together
$= 6! - 1 \times 5! \times 2!$
$= 480$

(d) (i) No. of committees $= {}^8C_3$
$= 56$

(ii) P(only Q'lders) $= \dfrac{{}^4C_3}{{}^8C_3}$
$= \dfrac{4}{56}$
$= \dfrac{1}{14}$