Sydney Girls High School

LABOR OMNIA VINCIT

2010
Extension 2 MATHEMATICS
[March 2010]

YEAR 12
ASSESSMENT TASK 2

Time Allowed: 90 minutes + 5 minutes reading time

Topics: Complex Numbers

Name: ____________________

Teacher: ____________________

General Instructions:

- There are THREE (3) Questions which are of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- Start each question on a new page.
- Write on one side of the paper only.
- Diagrams are NOT to scale.
- Board-approved calculators may be used.

$$\int \cos ax = \frac{1}{a}\sin ax + C$$

Total: 75 marks

Marks

QUESTION 1 (25 Marks)

a. Express $3cis\left(-\frac{\pi}{6}\right)$ in Cartesian form. 2

b. Solve $2x^2+3x+4=0$ giving your answer in the form $x+iy$. 2

c. If $z=3+2i$ and $w=5-i$, find in the form $x+iy$:

i. $w-2z$ 2

ii. wz 2

iii. $|w|$ 1

iv. $\bar{z}-\bar{w}$ 2

v. z^2+2z+1 2

d. Let $1, w, w^2$ be the cube roots of unity.

i. Show that $1+w+w^2=0$ 2

ii. Simplify $(1+w)^3(3w+3w^2)$ 2

e. Evaluate i^{3056} 1

Marks

f.

i. Solve $z^5 = 1$ over the complex field. 2

ii. Plot your solutions on an Argand diagram 2

g. Simplify $\dfrac{\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)}{\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}}$ giving your answer in the form $a+bi$ 3

QUESTION 2 (25 Marks)

a.

i. Find the exact value of the modulus and argument of $\dfrac{1+2i}{1-3i}$ 3

ii. Hence find the exact value of $\left(\dfrac{1+2i}{1-3i}\right)^9$ in Cartesian form. 2

b.

i. Find $\sqrt{-3-4i}$ 3

ii. Hence or otherwise, solve $z^2 - 3z + (3+i) = 0$ 2

Marks

c. The points A and B represent the complex numbers $4-i$ and $1+3i$ respectively.

i. Plot the points A and B on an Argand diagram and mark the point P so that $OAPB$ is a parallelogram. 2

ii. What complex number does P represent? 1

d. If z is such that $|z| = 3$ and $\arg z = \dfrac{2\pi}{3}$. Mark on the same Argand diagram:

i. z 1

ii. $\bar{z}$ 1

iii. $i\bar{z}$ 1

iv. $2z$ 1

v. $z - \bar{z}$ 1

e. The roots of the equation $z^6 - 1 = 0$ are: $1, -1,\ cis\dfrac{\pi}{3},\ \text{cis}\dfrac{2\pi}{3},\ \text{cis}\dfrac{-\pi}{3},\ \text{cis}\dfrac{-2\pi}{3}$. 2

Use this information to express $z^6 - 1$ as a product of real quadratic factors

f. Factorise $z^3 - 8$ over the complex field. 2

g. Find an expression for $\cos 5\theta$ in terms of powers of $\cos\theta$ 3

Marks

QUESTION 3 (25 Marks)

a. If $z = 1+\sqrt{3}i$ and $|w| = 4$ find:

i. $|z|$ 1

ii. the greatest value of $|z+w|$. 1

iii. if $|z+w|$ takes its maximum value, express w in the form $x+iy$ 1

b. If $z = \cos\theta + i\sin\theta$

i. Show that $z^n - \dfrac{1}{z^n} = 2i\sin(n\theta)$ 2

ii. Use the result obtained in i. to show that $\sin^4\theta = \dfrac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$ 2

iii. Hence evaluate $\int_0^{\frac{\pi}{2}} \sin^4 x\,dx$ 2

c. Find the equation of the locus of each of the following in Cartesian form.

i. $|z-1| = \mathrm{Re}(z)$ 2

ii. $|z+2-i| = 4$ 2

d. Sketch the locus of $|z-1| = |z+i|$ 2

Marks

e.

i. Sketch the locus given by $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{6}$ 2

ii. Find the cartesian equation of the locus found in part i. 2

f. Sketch the following regions on separate Argand diagrams.

i. $|z+2i| \le 3$ 3

ii. $|z-\bar{z}| < 2$ and $-\dfrac{\pi}{4} \le \arg z \le \dfrac{\pi}{4}$ 3

End of Task

Extension 2 Solutions:

Complex Numbers 2010

Question 1

a. $3\cos\left(-\frac{\pi}{6}\right)+3i\sin\left(-\frac{\pi}{6}\right)$

$=\frac{3\sqrt{3}}{2}-\frac{3}{2}i$

b. $x=\frac{-3\pm\sqrt{3^2-4\times2\times4}}{2\times2}$

$=\frac{-3\pm\sqrt{-23}}{4}$

$=\frac{-3\pm\sqrt{23}i}{4}$

c.

i. $5-i-2(3+2i)=-1-5i$

ii. $(3+2i)(5-i)=15-3i+10i+2$

$=17+7i$

iii. $\sqrt{5^2+(-1)^2}=\sqrt{26}$

iv. $3-2i-(5+i)=-2-3i$

v. $(z+1)^2=(3+2i+1)^2$

$=(4+2i)^2$

$=16+16i+4i^2$

$=12+16i$

d.

i. $w^3+w+w^2=w(w^2+1+w)$

$w\neq0$

$\therefore 1+w+w^2=0$

ii. $(-w^2)^3(-3)=-1\times-3$

$=3$

e.

$(i^4)^{764}=1^{764}$

$=1$

f.

i. $(r\,cis\theta)^5=1$

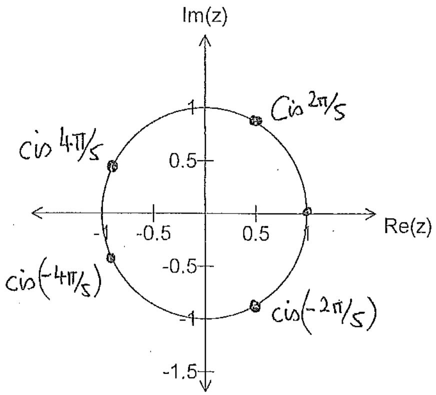
$r^5=1,\quad r=1$

$5\theta=0,2\pi,4\pi,6\pi,8\pi$

$\theta=0,\frac{2\pi}{5},\frac{4\pi}{5},-\frac{4\pi}{5},-\frac{2\pi}{5}$

$z=1, cis\frac{2\pi}{5}, cis\frac{4\pi}{5}, cis\left(-\frac{4\pi}{5}\right), cis\left(-\frac{2\pi}{5}\right)$

ii.



g. $\frac{cis\frac{23\pi}{20}}{cis\frac{2\pi}{3}}=cis\frac{29\pi}{60}$

$=\cos\frac{29\pi}{60}+i\sin\frac{29\pi}{60}$

$=0.05233595624+0.9986295348i$

Question 2

a. i. $\frac{1+2i}{1-3i}\times\frac{1+3i}{1+3i}=\frac{1+5i+6i^2}{1-9i^2}$

$=-\frac{1}{2}+\frac{1}{2}i$

mod $=\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$

arg $=\frac{3\pi}{4}$

a.ii. $\left(\frac{\sqrt{2}}{2}\text{cis}\frac{3\pi}{4}\right)^9=\frac{\sqrt{2}}{32}\text{cis}\left(\frac{27\pi}{4}\right)$

$=-\frac{1}{32}+\frac{1}{32}i$

b.i. $\sqrt{-3-4i}=x+iy$

$x^2-y^2=-3$ and $2xy=-4$

$x^4+3x^2-4=0$

$(x^2+4)(x^2-1)=0$

$x=\pm1,\quad y=\mp2$

$\sqrt{-3-4i}=\pm(1-2i)$

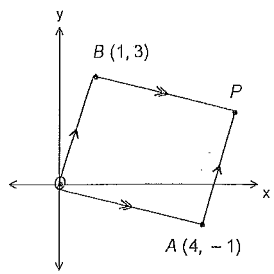
b.ii. $z=\frac{3\pm\sqrt{9-4(3+i)}}{2}$

$=\frac{3\pm\sqrt{-3-4i}}{2}$

$=\frac{3\pm(1-2i)}{2}$

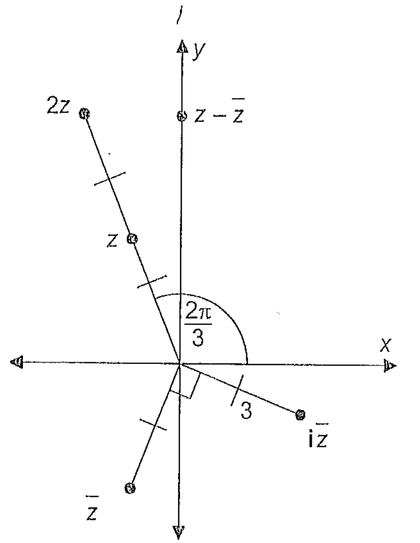
$z=2-i$ or $1+i$

c.i.



c.ii. $5+2i$

d.



e. $(z-1)(z+1)\left(z-cis\frac{\pi}{3}\right)\left(z-cis\frac{2\pi}{3}\right)$

$\left(z-cis\left(-\frac{\pi}{3}\right)\right)\left(z-cis\left(-\frac{2\pi}{3}\right)\right)$

$=(z^2-1)\left(z^2-\left(2\cos\frac{\pi}{3}\right)z+1\right)\left(z^2-\left(2\cos\frac{\pi}{3}\right)2z+1\right)$

$=(z^2-1)(z^2-z+1)(z^2+z+1)$

f. $(z-2)(z^2+2z+1-3i^2)$

$=(z-2)\left((z+1)^2-3i^2\right)$

$=(z-2)(z+1+\sqrt{3}i)(z+1-\sqrt{3}i)$

g. $(\cos\theta+\sin\theta)^5=c^5+5c^4is+10c^3i^2s^2+10c^2i^3s^3$

$+5ci^4s^4+i^5s^5$

Equate real parts

$\cos5\theta=c^5-10c^3s^2+5cs^4$

$=c^5-10c^3(1-c^2)+5c(1-c^2)^2$

$=16\cos^5\theta-20\cos^3\theta+5\cos\theta$

Question 3

a.

i. $|z| = \sqrt{1^2 + \sqrt{3}^2}$

$= 2$

ii. $|z| + |w| = 2 + 4$

$= 6$

iii. if $|z + w|$ takes its maximum value, then

$\arg(z) = \arg(w)$.

$\therefore\ w = 2z$

$w = 2(1 + \sqrt{3}i)$

$= 2 + 2\sqrt{3}i$

b.

i. $z = \cos\theta + i\sin\theta$

$z^n = \cos(n\theta) + i\sin(n\theta)$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos(n\theta) - i\sin(n\theta)$

$z^n - \dfrac{1}{z^n} = \cos(n\theta) + i\sin(n\theta) - (\cos(n\theta) - i\sin(n\theta))$

$= 2i\sin n\theta$

ii. from i. $z - \dfrac{1}{z} = 2i\sin\theta$

$(2i\sin\theta)^4 = \left(z - \dfrac{1}{z}\right)^4$

$16\sin^4\theta = z^4 - 4z^3\cdot\dfrac{1}{z} + 6z^2\cdot\dfrac{1}{z^2} - 4z\cdot\dfrac{1}{z^3} + \dfrac{1}{z^4}$

$= \left(z^4 + \dfrac{1}{z^4}\right) - 4\left(z^2 + \dfrac{1}{z^2}\right) + 6$

$= 2\cos 4\theta - 8\cos 2\theta + 6$

$\sin^4\theta = \dfrac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$

iii. $\int_0^{\frac{\pi}{2}} \sin^4 x\,dx = \dfrac{1}{8}\int_0^{\frac{\pi}{2}} (\cos 4x - 4\cos 2x + 3)dx$

$= \dfrac{1}{8}\left[\dfrac{1}{4}\sin 4x - 2\sin 2x + 3x\right]_0^{\frac{\pi}{2}}$

$= \dfrac{1}{8}\left(\dfrac{3\pi}{2} - 0\right)$

$= \dfrac{3\pi}{16}$

c.

i. $|z - 1| = \text{Re}(z)$

$|x + iy - 1| = x$

$\sqrt{(x-1)^2 + y^2} = x$

$(x-1)^2 + y^2 = x^2$

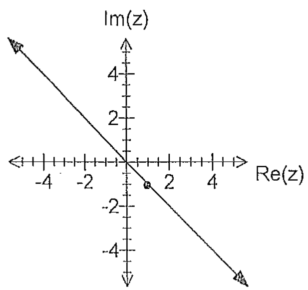
$y^2 = 2x - 1$

ii. $|z + 2 - i| = 4$

circle centre $(-2,1)$ radius 4

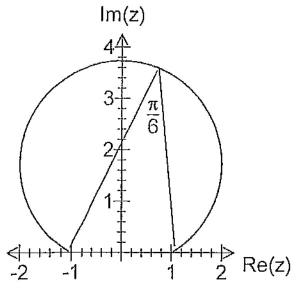
$(x+2)^2 + (y-1)^2 = 16$

d.

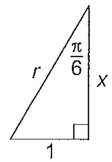


e.

i.



ii.



$\tan\dfrac{\pi}{6} = \dfrac{1}{x}$

$x = \sqrt{3}$

$\therefore$ centre of circle is $(0, \sqrt{3})$

By Pythagoras:

$1 + \sqrt{3}^2 = r^2$

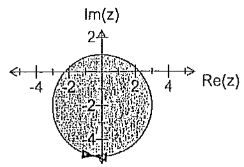
$r = 2$

$\therefore$ equation is: $x^2 + (y - \sqrt{3})^2 = 4$

f.

i.

$|z + 2i| \le 3$



ii.

$|z + \bar{z}| < 2$ and $-\dfrac{\pi}{4} \le \arg z \le \dfrac{\pi}{4}$