ASCHAM - 2011

Year 12 Extension 1 Mathematics 25th March 2011 1 period = 40 minutes
HSC Assessment 2 on the Trig Functions, Inverse Trig Functions,
Further Trig Methods and Circle Geometry
Total = 33 marks
Pender Year 11 Ch 14, Year 12 Ch 1 & 2 and Ch 9
Use can use the table of Standard Integrals provided

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PART A (11 marks)

1. a) Express $\frac{7\pi}{12}$ in degrees. **1**

   b) Evaluate $\lim_{x\to 0}\frac{\left(\tan\frac{x}{2}\right)}{3x}$ **1**

2. Differentiate

   a) $y=\sin^{-1}(5x-3)$ **2**

   b) $y=x^2\tan^{-1}(3x)$ **2**

3. By using the substitution $t=\tan\theta$, or otherwise, show $\frac{1-\cos 2\theta}{1+\cos 2\theta}=\tan^2\theta$ **2**

4. (i) By expanding the LHS, show $\sin(3x+2x)+\sin(3x-2x)=2\sin 3x\cos 2x$
   1

   (ii) Hence or otherwise find $\frac{d}{dx}(\sin 3x\cos 2x)$ **2**

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PART B (11 marks)

5. (i) State the domain and range of $y=5\cos^{-1}(\frac{x}{4})$ **2**

   (ii) Hence or otherwise, sketch this curve. **1**

   (iii) Find the area bounded by the curve $y=5\cos^{-1}(\frac{x}{4})$ and the x and y axes. **2**

6. Find the following integrals:

   (a) $\int \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)dx$ **2**

   (b) (i) Show $\int_{-\frac{1}{4}}^{\frac{3}{5}}\frac{1}{1+x^2}dx=\tan^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{-1}{4}\right)$ **1**

   (ii) Then show $\tan^{-1}(\frac{3}{5})-\tan^{-1}(\frac{-1}{4})=\frac{\pi}{4}$ **3**

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PART C (11 marks)

7. (i) Write $4\sin x-3\cos x$ in the form $R\sin(x-\alpha)$ where $R>0$ and $0<\alpha<\frac{\pi}{2}$ **2**

   (ii) Hence find the **general** solution of $4\sin x-3\cos x=5$
   3

....

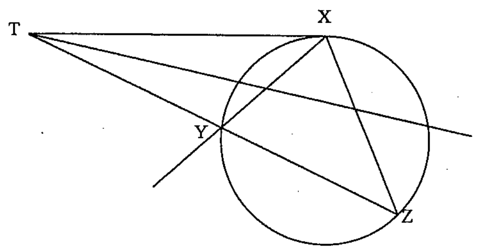
Tear off Q9 and staple it to your answer page for PART C

8. In the diagram below the points X, Y and Z lie on the circle and ZY produced meets the tangent from X at the point T. The bisector of the angle XTZ intersects XY and XZ at E and F respectively. Let $\angle TXY=\alpha$

   (i) Using the diagram provided mark the angle α and points E and F.
   Show all necessary working and angles on your diagram. **2**

   (ii) Explain why $\angle XZY=\alpha$
   1

   (ii) Hence prove that ΔXEF is isosceles. **3**



ASCHAM - YR 12 - EXT 1. TASK 2 - SOLUTIONS.

1a) $\frac{7\pi}{12} = 105°$ $\quad \frac{7\pi}{12} \times \frac{180}{\pi} = 105°$

b) $\lim_{x \to 0} \frac{\tan \frac{x}{2}}{3x} = \lim_{x \to 0} \frac{\frac{1}{2} \tan \frac{x}{2}}{3 \cdot \frac{1}{2}x}$ $\quad \lim_{x \to 0} \frac{\frac{1}{2}x \tan \frac{x}{2}}{3 \cdot \frac{1}{2}x}$

$= \frac{1}{6}.$

2a) $y = \sin^{-1}(5x-3)$

$y' = \frac{5}{\sqrt{1-(5x-3)^2}}$

$= \frac{5}{\sqrt{1-25x^2+30x-9}}$

$= \cancel{\frac{5}{\sqrt{10+30x-25x^2}}}$ $\quad \frac{5}{\sqrt{30x-8-25x^2}}$

b) $y = x^2 \tan^{-1}(3x)$

$y' = 2x\tan^{-1}3x + \frac{x^2 \times 3}{1+9x^2}$

$= \frac{3x^2 + 2x\tan^{-1}3x}{1+9x^2}$ ✗

$\frac{1}{2}$

3. $\tan\theta = t$

$\frac{1-\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta$

LHS $= \frac{1 - \frac{1-t^2}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}}$

$= \frac{1+t^2-1+t^2}{1+t^2+1-t^2}$

$= \frac{2t^2}{2}.$

$= t^2 = \text{RHS}$

$= \tan^2\theta$ as required

4 i) LHS $= \sin(3x+2x) + \sin(3x-2x)$

$= \sin 3x\cos 2x + \cancel{\sin 2x \cos 3x} + \sin 3x\cos 2x - \cancel{\sin 2x\cos 3x}$

$= 2\sin 3x\cos 2x$

$=$ RHS

as required.

ii) $\frac{d}{dx}(\sin 3x\cos 2x) = \frac{d}{dx}\frac{1}{2}(\sin(3x+2x) + \sin(3x-2x))$.

$= \frac{d}{dx}(\sin 5x + \sin x)\frac{1}{2}.$

$= \frac{5\cos 5x + \cos x}{2}.$

$y = x^2\tan^{-1}(3x)$

$= 2x\tan^{-1}(3x) + \frac{3 \times x^2}{1+9x^2}$

$= \frac{2x\tan^{-1}(3x) + 18x^3\tan^{-1}(3x) + 3x^2}{1+9x^2}.$

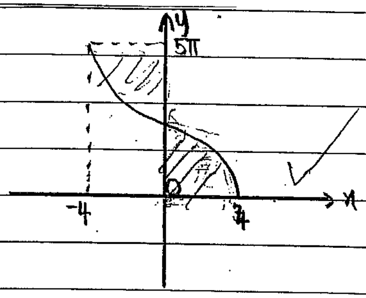
5. i) $y = 5\cos^{-1}(\frac{x}{4})$.

Domain: $-1 \le \frac{x}{4} \le 1$.

$-4 < x \le 4$.

Range: $0 \le y \le \cancel{10\pi}\ 5\pi$.

ii)



iii) Area.

$y = 5\cos^{-1}\frac{x}{4}$.

$x = 5\cos^{-1}\frac{y}{4}$.

$\frac{x}{5} = \cos^{-1}\frac{y}{4}$

$\frac{y}{\cancel{4}} = \cos^{-1}\frac{x}{5}$.

$y = 4\cos\frac{x}{5}$

By symmetry.

Area $= 4\int_0^{2\frac{1}{2}\pi} 4\cos\frac{x}{5}\,dx$

$= 4[20\sin\frac{x}{5}]_0^{2\frac{1}{2}\pi}$

$= \cancel{(40(\sin 0 - \sin\frac{\pi}{2}))}$ $\quad 40\sin\frac{2\frac{1}{2}\pi}{5} - 40\sin 0$

$= 40u^2$

6 a) $\int \sin\frac{x}{2}\cos\frac{x}{2}\,dx = \frac{1}{2}\int 2\sin\frac{x}{2}\cos\frac{x}{2}\,dx$

$= \frac{1}{2}\int \sin x\,dx$.

$= -\frac{1}{2}\cos x + C$.

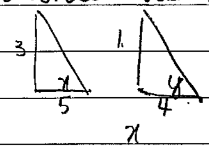
b) i) $\int_{-\frac{1}{4}}^{\frac{3}{5}} \frac{1}{1+x^2}\,dx = [\tan^{-1}x]_{1/4}^{3/5}$

$= \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{1}{4}$

as required.

ii) $\tan^{-1}\frac{3}{5} - \tan^{-1}\frac{1}{4}$ let $\tan^{-1}\frac{3}{5} - \tan^{-1}\frac{1}{4} = x - y$.

consider $\tan(x-y) = \cancel{\frac{\tan x + \tan y}{1-\tan x\tan y}} = \frac{\tan x - \tan y}{1+\tan x\tan y}$



$= \cancel{\frac{\frac{3}{5}+\frac{1}{4}}{1-\frac{3}{20}}} = \frac{\frac{3}{5}-\frac{1}{4}}{1+\frac{3}{20}}$

$\tan^{-1}\frac{3}{5} = x \quad \tan^{-1}\frac{1}{4} = y \qquad = \cancel{\frac{12+5}{20-3}} = \frac{12-5}{20+3} = \frac{7}{7}$

$\tan x = \frac{3}{5} \quad \tan y = \frac{1}{4} \qquad = \cancel{\frac{17}{17}} \qquad = 1$

$x - y = \frac{\pi}{4}$

$\therefore \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{1}{4} = \frac{\pi}{4}$ as required.

~~tan(x-~~

$\tan^{-1}\frac{3}{5} - \tan^{-1}(-\frac{1}{4})$.

$\tan(x-y) = \cancel{\frac{\tan\frac{3}{5}+\tan\frac{1}{4}}{1+\frac{3}{20}}} \quad \frac{x-y}{1+\tan x\tan y}$

$= \cancel{\frac{\frac{3}{5}+\frac{1}{4}}{1+\frac{3}{20}}} \quad \frac{\frac{3}{5}+\frac{1}{4}}{1-\frac{3}{20}}$

$= \frac{12+5}{20-3} = \frac{17}{17} = 1$

$\therefore \tan^{-1}\frac{3}{5} - \tan^{-1}(-\frac{1}{4}) = x - y$

$\tan(x-y) = 1$

$x - y = \frac{\pi}{4}$

$\therefore \tan^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4} = \frac{\pi}{4}$

7 i) $4\sin x - 3\cos x = 5\sin x\cos\alpha - 5\cos x\sin\alpha$.

$5\cos\alpha = 4$ $\quad$ $5\sin\alpha = 3\cancel{\cos x}$

$\cos\alpha = \frac{4}{5}$ $\quad$ $\sin\alpha = \frac{3}{5}$

$\alpha = \cos^{-1}\frac{4}{5}$ $\quad$ $\alpha = \sin^{-1}\frac{3}{5}$.

2 $\quad 5\sin(x - \sin^{-1}\frac{3}{5}) = 4\sin x - 3\cos x$.

ii) $5\sin(x - \sin^{-1}\frac{3}{5}) = 5$

$x - \sin^{-1}\frac{3}{5} = n\pi + (-1)^n\frac{\pi}{2}$.

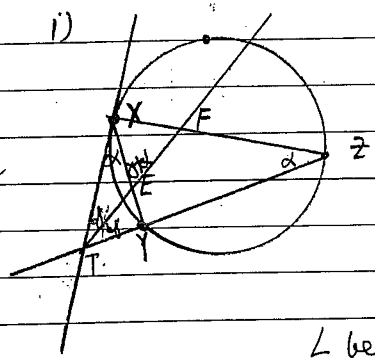
$x = \sin^{-1}\frac{3}{5} + n\pi + (-1)^n\frac{\pi}{2}$.

3 $\quad \doteqdot 0.64350 + n\pi + (-1)^n\frac{\pi}{2}$.

$= 0.6435 + n\pi + (-1)^n\frac{\pi}{2}$ (to 4 dp).

8.

i)



2

∠ between tangent + chord = $\quad -\frac{1}{2}$

1/2 ii) $\angle XZY = \alpha$. (∠ in alternate segment)

iii) $\triangle XEF$

Let $\angle XTE = \angle ETY = \gamma$ (bisected).

$\angle XEF = \gamma + \alpha$ (exterior ∠ sum of $\triangle XET$).

$\angle XFE = \gamma + \alpha$ (exterior ∠ sum of $\triangle TFZ$)

3 $\quad$ ∠ : $\angle XEF$

$\therefore$ $\triangle XEF$ is isosceles (base ∠s are equal).