THE SCOTS COLLEGE

Extension 1 Mathematics

Pre-Trial Examination

17th March 2011

Time Allowed: 90 minutes + 5 minutes

Instructions:

- Show all necessary workings
- Approved non-programmable calculators may be used
- Begin a new sheet of paper for each question
- A removable page with standard integrals is located at the back

Outcomes to be assessed:

Preliminary	Q1	/10
Functions	Q2, Q3, Q4	/34
Calculus	Q5, Q6	/22
	TOTAL	/66

Question One (10 marks) Begin a new sheet of paper

a) Solve $\frac{2x+6}{x-3} < -2$ *(2)*

b) P and Q are the end points of a focal chord of the parabola $x^2 = 4ay$ with focus S. If the co-ordinates of P and Q are $(2ap, ap^2)$ and $(2aq, aq^2)$ respectively,

i) Write down the gradients of PS and QS. *(1)*

ii) Show that $pq = -1$ *(1)*

iii) Find the coordinates of the midpoint R of PQ in terms of p *(2)*

iv) Show that the equation of the locus of R is $x^2 = 2a(y - a)$ *(2)*

c) Let $f(x) = \log_e[(x-2)(1+x)]$, what is the domain of $f(x)$ *(2)*

Question Two (10 marks) Begin a new sheet of paper

a) Given that $(x + 2)$ and $(x - 3)$ are factors of $P(x) = x^3 - 6x^2 + px + q$, find the values of p and q. *(3)*

b) Without the use of calculus, sketch the graph of $f(x) = x(2x - 1)(x - 1)^3$, clearly showing any axis intercepts. *(3)*

c) Solve the equation $3x^3 - 7x^2 - 70x + 24 = 0$, given that the product of two of the roots is 2. *(4)*

Question Three (6 marks) Begin a new sheet of paper

Given the function $y = \frac{\sin x}{x-3}$

a) Show that a root of $\frac{\sin x}{x-3} = 0$ lies between $x = 6$ and $x = 6.5$. *(2)*

b) Use the method of halving the interval to show that this root lies between $x = 6.25$ and $x = 6.375$. *(2)*

c) Use one application of Newton's method to find an approximation of this root correct to 3 decimal places using a first approximation of *6.25* *(2)*

QUESTION FOUR (18 MARKS) BEGIN A NEW SHEET OF PAPER

a) Given the function $f(x) = x^2 - 4x$

i) Graph the function $y = x^2 - 4x$. (2)

ii) State the largest positive domain for *f(x)* that will allow you to define the inverse function. (1)

iii) Find the inverse function and state the domain and range of the inverse function (3)

iv) Calculate the exact value of $f^{-1}(2)$ (1)

v) Graph the inverse function from *(iii)* on the same axes used in part *(i)* (2)

vi) Calculate the point of intersection of $f(x)$ and $f^{-1}(x)$ (2)

b) Evaluate $\tan^{-1}(-\sqrt{3})$ (1)

c) Prove that $\sin^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$ (2)

d) Given $y = a\cos^{-1}(bx)$ is a function. (where $a, b > 0$)

i) State the domain and range (2)

ii) Sketch this function (2)

QUESTION FIVE (8 MARKS) BEGIN A NEW SHEET OF PAPER

a) Given $f(x) = e^{5x+1}$

i) Find $f'(x)$ (1)

ii) Find $f^{-1}(x)$ (2)

iii) Differentiate $f^{-1}(x)$ (1)

iv) Deduce that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$ (1)

b) Find the equation of the normal to the curve $y = \tan^{-1}x$ at the point $x = \sqrt{3}$. (3)

QUESTION SIX (14 MARKS) BEGIN A NEW SHEET OF PAPER

a) Differentiate, $e^{(\cos^{-1}x)}$ (2)

b) $\int \frac{2}{\sqrt{25 - 4x^2}}\,dx$ (3)

c) Find the area enclosed between the curve $y = \cos^{-1}x$, the y-axis and the lines $y = 0$ and $y = \frac{\pi}{4}$ (2)

d) For the curve $y = \frac{1}{\sqrt{1+x^2}}$, find the volume of the solid formed by rotating the curve about the *x-axis* from $x = \frac{1}{\sqrt{3}}$ to $x = 1$ (4)

e) Find $\int \frac{-x^2}{\sqrt{1-x^6}}\,dx$ by using the substitution $u = x^3$ (3)

END OF EXAM

ANSWER SHEET

Name: Ext 1 - Pre-trial 2010

Teacher: SOLUTIONS

Question No. 1

a) $\dfrac{2x+6}{x-3} < -2$

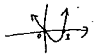
$x \neq 3$

$(2x+6)(x-3) < -2(x-3)^2$

$(2x+6)(x-3)+2(x-3)^2 < 0$

$(x-3)(2x+6+2x+6) < 0$

$(x-3)(4x) < 0$



$0 < x < 3$

b) i) $M_{PS} = \dfrac{ap^2-a}{2ap}$

$M_{RQ} = \dfrac{aq^2-a}{2ap}$

ii) PQ, PS, RS are collinear

$\therefore M_{PS} = M_{PQ}$

$\dfrac{ap^2-a}{2ap} = \dfrac{aq^2-a}{2aq}$

$\dfrac{p^2-1}{p} = \dfrac{q^2-1}{q}$

$p^2q - q = q^2p - p$

$p^2q - q^2p = q - p$

$pq(p-q) = -(p-q)$

$pq = -1$

ANSWER SHEET

Name:

Teacher:

Question No. Q1

b) iii) $R = \left(\dfrac{2ap+2aq}{2}, \dfrac{ap^2+aq^2}{2}\right)$

$= \left[a(p+q), \dfrac{a}{2}(p^2+q^2)\right]$

Note: $pq = -1 \iff q = \dfrac{-1}{p}$

$R \left[a\left(p+\dfrac{-1}{p}\right), \dfrac{a}{2}\left(p^2+\dfrac{1}{p^2}\right)\right]$

iv) $x = a\left(p-\dfrac{1}{p}\right)$ ...①

$y = \dfrac{a}{2}\left(p^2+\dfrac{1}{p^2}\right)$ ...②

$\therefore$ From ①

$x^2 = a^2\left(p-\dfrac{1}{p}\right)^2$

$= a^2\left(p^2+\dfrac{1}{p^2}-2\right)$

also from ②

$\dfrac{2y}{a} = p^2 + \dfrac{1}{p^2}$

substituting

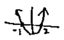
$x^2 = a^2\left(\dfrac{2y}{a}-2\right)$

$x^2 = 2ay - 2a^2$

$x^2 = 2a(y-a)$

c) $f(x) = \ln[(x-2)(1+x)]$

D: $(x-2)(1+x) > 0$



D: $x > 2$ & $x < -1$

ANSWER SHEET

Name:

Teacher:

Question No. Q2

a) $P(-2) = 0$

$0 = (-2)^3 - 6(-2)^2 + p(-2) + q$

$32 = -2p + q$

$32 + 2p = q$ ...①

$P(3) = 0$

$0 = 27 - (6 \times 9) + 3p + q$

$27 = 3p + q$ ...②

sub in ① into ②

$27 = 3p + 32 + 2p$

$-5 = 5p$

$-1 = p$

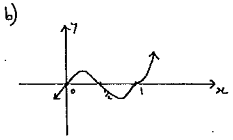
$\therefore$ From ①

$32 + 2(-1) = q$

$30 = q$

$p = -1$ & $q = 30$

b)



c) Roots are α, β, γ

$\alpha\beta = 2$ (given)

$\alpha\beta\gamma = \dfrac{-24}{3}$

$2\gamma = \dfrac{-24}{3}$

$\gamma = -4$

$\alpha\beta + \alpha\gamma + \beta\gamma = \dfrac{-70}{3}$

$2 - 4\alpha - 4\beta = \dfrac{-70}{3}$

$\alpha + \beta = 19/3$

Note $\alpha\beta = 2 \iff \alpha = \dfrac{2}{\beta}$

$\dfrac{2}{\beta} + \beta = \dfrac{19}{3}$

$3\beta^2 - 19\beta + 6 = 0$

$(3\beta - 1)(\beta - 6) = 0$

$\therefore \beta = \frac{1}{3}$ or $\beta = 6$

Hence

$\alpha = 6$

$\beta = \frac{1}{3}$

$\gamma = -4$

ANSWER SHEET

Name:

Teacher:

Question No. 3

a) $f(6) = \dfrac{\sin(6)}{3} = -0.0931 < 0$

$f(6.5) = \dfrac{\sin(6.5)}{3.5} = 0.06146 > 0$

$\therefore$ a root exist given a sign change from $x=6$ to $x=6.5$

b)

$f\left(\dfrac{6+6.5}{2}\right) = f(6.25) = -0.0102 < 0$

$\therefore$ exists $6.25 < x < 6.5$

$f\left(\dfrac{6.25+6.5}{2}\right) = f(6.375) = 0.2117 > 0$

$\therefore$ root exists

$6.25 < x < 6.375$

c) $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

let $x_0 = 6.25$

$f(x) = \dfrac{\sin x}{x-3}$

$f'(x) = \dfrac{-\sin x}{(x-3)^2} + \dfrac{\cos x}{x-3} = \dfrac{-\sin x + (\cos x)(x-3)}{(x-3)^2}$

$\therefore$

$x_1 = 6.25 - \left(\dfrac{-0.0102}{0.31066}\right)$

$= 6.283$

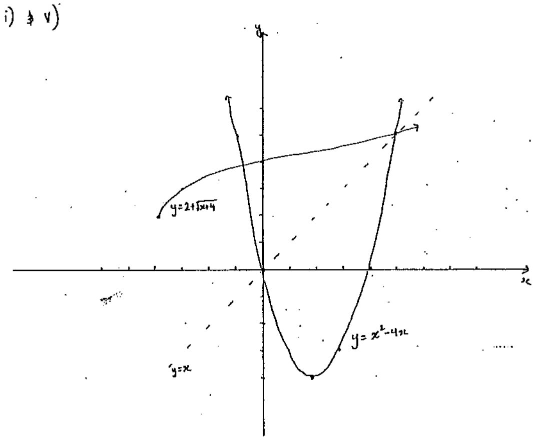
ANSWER SHEET

Name:

Teacher:

Question No. 4

i) & v)



ii) D: $x \geq 2$

ANSWER SHEET

Name:

Teacher:

Question No. 4

iii) $x = y^2 - 4y$

$= (y-2)^2 - 4$

$x + 4 = (y-2)^2$

$y - 2 = +\sqrt{x+4}$ (only +ve square root to be a fn)

$y = 2 + \sqrt{x+4}$

$f^{-1}(x) = 2 + \sqrt{x+4}$

For $f^{-1}(x)$

D: $x \geq -4$

R: $y \geq 2$

iv) $f^{-1}(2) = 2 + \sqrt{6}$

v) See graph.

vi) $f(x)$ and $f^{-1}(x)$ meet on the line $y = x$.

$\therefore$ intersection

$x^2 - 4x = x$ or $2 + \sqrt{x+4} = x$

$x^2 - 5x = 0$

$x(x-5) = 0$

$x = 0$ (outside domain)

or

$x = 5$

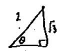
Harder to solve!

b) $\tan^{-1}(-\sqrt{3})$

$\therefore \tan^{-1}(-\sqrt{3}) = -\dfrac{\pi}{3}$

Note: Domain of $\tan^{-1}x$;

$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$



$\theta = 60$

$\theta = \dfrac{\pi}{3}$

ANSWER SHEET

Name:

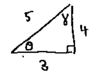
Teacher:

Question No. 4

c)

$\sin^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{4}{5}\right) =$

$\theta + \gamma =$



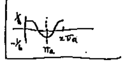
$90° = \gamma + \theta = \dfrac{\pi}{2}$

$\therefore$ By inspection

$\sin^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{4}{5}\right) = \dfrac{\pi}{2}$

d) i) $y = a\cos^{-1}(bx)$ $(a, b > 0)$

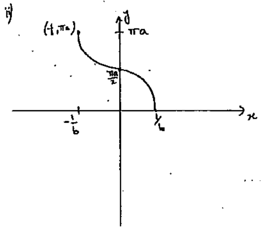
$x = \frac{1}{b}\cos\left(\frac{y}{a}\right)$



D: $-\frac{1}{b} \leq x \leq \frac{1}{b}$

R: $0 \leq y \leq \pi a$

ii)



ANSWER SHEET

Name:

Teacher:

Question No. 5

a) i) $f'(x) = 5e^{5x+1}$

ii) $x = e^{5y+1}$

$\log_e x = 5y + 1$

$\frac{1}{5}\ln(x) - \frac{1}{5} = y$

$f^{-1}(x) = \frac{1}{5}\ln(x) - \frac{1}{5}$

iii) $y = \frac{1}{5}\ln(x) - \frac{1}{5}$

$\dfrac{dy}{dx} = \dfrac{1}{5x}$

iv) $\dfrac{dy}{dx} \times \dfrac{dx}{dy}$

using $y = e^{5x+1}$

$\dfrac{dy}{dx} = 5e^{5x+1}$

$\dfrac{dx}{dy} = \dfrac{1}{5y} = \dfrac{1}{5e^{5x+1}}$

$\dfrac{dy}{dx} \times \dfrac{dx}{dy}$

$5e^{5x+1} \times \dfrac{1}{5e^{5x+1}} = 1$

ANSWER SHEET

Name: ____________

Teacher: ____________

Question No. 5

b) $y = \tan^{-1}(x) = f(x)$

$\frac{dy}{dx} = \frac{1}{1+x^2} = f'(x)$

$f(\sqrt{3}) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$f'(\sqrt{3}) = \frac{1}{1+3} = \frac{1}{4}$

$\therefore$ Normal gradient is $\frac{-1}{f'(\sqrt{3})} = -4$

Equation of normal

$y - y_1 = m(x - x_1)$

$y - \frac{\pi}{3} = -4(x - \sqrt{3})$

$4x + y - \frac{\pi}{3} - 4\sqrt{3} = 0$

$12x + 3y - \pi - 12\sqrt{3} = 0$

ANSWER BOOKLET

Name: ____________

Teacher: ____________

Question No. 6

a) $y = e^{\cos^{-1}(x)}$

$y = e^u$ let $u = \cos^{-1}(x)$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{du} = e^u$

$\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$

$\frac{dy}{dx} = e^u \times \frac{-1}{\sqrt{1-x^2}}$

$= \frac{-e^{\cos^{-1}(x)}}{\sqrt{1-x^2}}$

b) $\int \frac{2}{\sqrt{25-4x^2}}\,dx$

$\int \frac{2}{5\sqrt{1-\frac{4x^2}{25}}}\,dx$

let $u^2 = \frac{4x^2}{25}$

$u = \frac{2x}{5}$

$\frac{du}{dx} = \frac{2}{5}$

$dx = \frac{5}{2}\,du$

$\int \frac{2}{5} \times \frac{1}{\sqrt{1-u^2}} \times \frac{5}{2}\,du$

$\int \frac{1}{\sqrt{1-u^2}}\,du$

$\sin^{-1}(u) + C$

$\sin^{-1}\left(\frac{2x}{5}\right) + C$

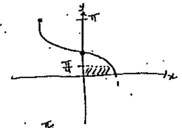
Question No. Q6

c) $y = \cos^{-1}(x)$

$y = 0$

$y = \frac{\pi}{4}$

$x = 0$



Area $= \int_0^{\pi/4} \cos(x)\,dx$

$= \left[\sin(x)\right]_0^{\pi/4}$

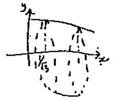
$= \sin(\frac{\pi}{4}) - \sin(0)$

$= \frac{1}{\sqrt{2}} - 0$

Area $= \frac{1}{\sqrt{2}}$ sq units.

$= \frac{\sqrt{2}}{2}$ sq units

d) $\frac{1}{\sqrt{1+x^2}}$



$V = \pi \int_{1/\sqrt{3}}^{1} [f(x)]^2\,dx$

$= \pi \int_{1/\sqrt{3}}^{1} \frac{1}{1+x^2}\,dx$

$= \pi \left[\tan^{-1}(x)\right]_{1/\sqrt{3}}^{1}$

$= \pi \left[\tan^{-1}(1) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$

$= \pi \left[\frac{\pi}{4} - \frac{\pi}{6}\right]$

$= \frac{\pi^2}{4} - \frac{\pi^2}{6}$

$= \frac{3\pi^2 - 2\pi^2}{12}$

$V = \frac{\pi^2}{12}$

ANSWER SHEET

Name: ____________

Teacher: ____________

Question No. 6

e) $\int \frac{-x^2}{\sqrt{1-x^6}}\,dx$

let $u = x^3$

$\frac{du}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$

$\int \frac{-1}{\sqrt{1-u^2}} \times x^2 \times \frac{du}{3x^2}$

$\frac{1}{3}\int \frac{-1}{\sqrt{1-u^2}}\,du$

$\left(\frac{1}{3}\left[\cos^{-1}(u)\right]\right) + C$

$\frac{1}{3}\cos^{-1}(x^3) + C$