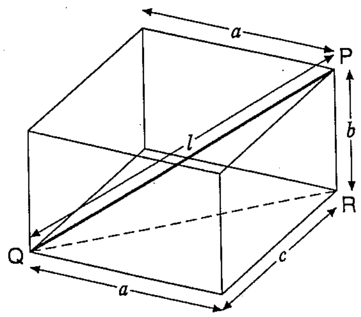
Pythagoras and trigonometry in three dimensions

Pythagoras in three dimensions

Problems in three dimensions can usually be broken down into two-dimensional parts.
But if you have to find the diagonal length of a cuboid (l in the diagram), you will find it useful to know that

$$l^2 = a^2 + b^2 + c^2$$

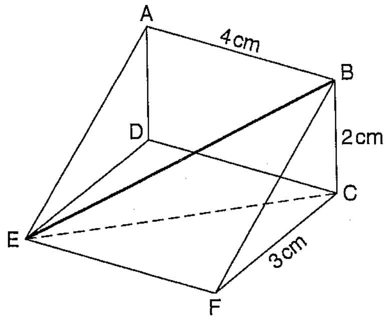
$QR^2 = a^2 + c^2$
$QP^2 = QR^2 + b^2 = a^2 + b^2 + c^2$



Angles in three dimensions

To find the angle between a line and a plane, you need to find a triangle which is perpendicular to the plane and has the line as one of its sides.

Example
The diagram shows a triangular prism. Angle FCB = 90°.
Work out
(a) the length of EB,
(b) the angle EB makes with the plane DEFC.

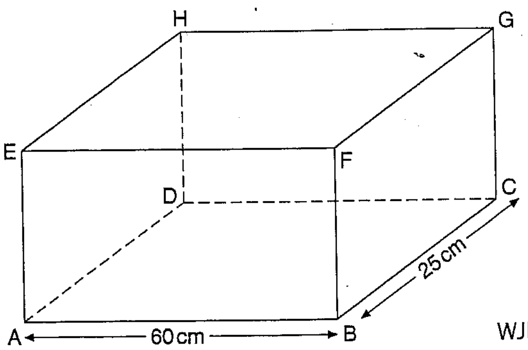


(a) $EC^2 = EF^2 + FC^2 = 4^2 + 3^2 = 16 + 9 = 25$
$EC = \sqrt{25} = 5$
Now triangle ECB is right-angled.
So $EB^2 = EC^2 + CB^2 = 5^2 + 2^2 = 25 + 4 = 29$.
$EB = \sqrt{29} = 5{\cdot}385\ldots = 5{\cdot}4\,\text{cm}$ (to 2 s.f.)
(Note that you could go straight to this answer, since EB is the diagonal of a cuboid with sides 2, 3 and 4.
So $EB^2 = 2^2 + 3^2 + 4^2 = 4 + 9 + 16 = 29$, $EB = \sqrt{29} = 5{\cdot}4\,\text{cm}$ (to 2 s.f.)

(b) The angle between the line EB and the plane DEFC is ∠BEC, since triangle BEC is perpendicular to the plane.
$\angle ECB = 90°$, so $\sin BEC = \frac{BC}{EB} = \frac{2}{\sqrt{29}}$
$\angle BEC = 21{\cdot}801\ldots° = 22°$ (to 2 s.f.)

Always use unrounded answers if you need to use a length or angle again.

1 In the rectangular box shown, AB is 60 cm and BC is 25 cm.
The length of diagonal AG is 72 cm.
(a) Calculate the length of AC.
(b) Calculate the height of the box.
(c) Calculate the angle that AG makes with the face ADHE.

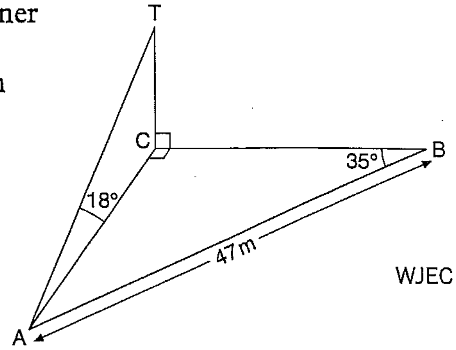


WJEC

2 A telegraph pole, CT, stands vertically upright in one corner of a flat triangular field, ABC, as shown in the diagram.
The field is right-angled at C, ∠ABC is 35° and the length of side AB is 47 m.
The angle of elevation of T from A is 18°.
Calculate

(a) the length of CA,

(b) the height of the telegraph pole.



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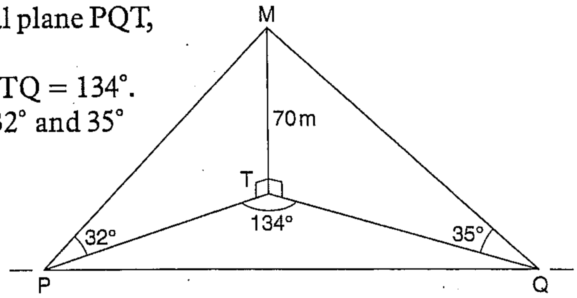
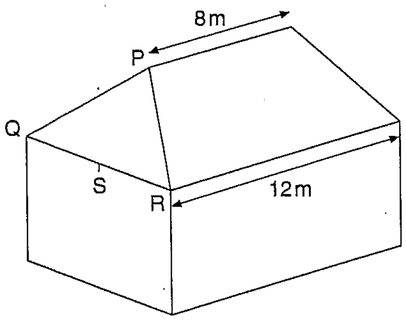
3 Peter and Queenie are surveying on a horizontal plane PQT, using PQ as a base line.
TM is a television mast of height 70 m. Angle PTQ = 134°.
The angles of elevation of M from P and Q are 32° and 35° respectively.

(a) Show that the distance from P to Q is approximately 195 m.

(b) Calculate the size of angle TPQ.



MEG/ULEAC (SMP)

4 

The roof of a house has a top ridge of length 8 m as shown.
The length of the house is 12 m.
The height of the ridge above the base of the roof is 1·8 m.
All of the sloping edges of the roof are the same length as each other. S is the midpoint of the edge QR.

Calculate the angle that PS makes with the horizontal.

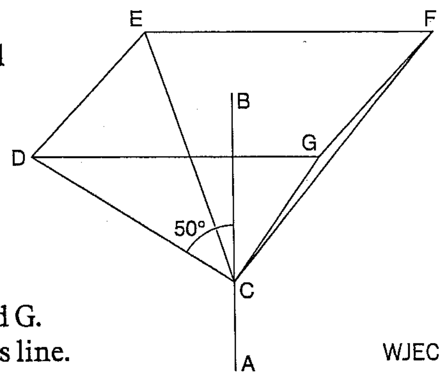
MEG/ULEAC (SMP)

5 The frame of a rotary garden clothes line consists of a vertical pole, AB, with four identical arms CD, CE, CF and CG each hinged to AB at a point C. The point C is 0·7 m below B. The arms are inclined at 50° to the vertical and their ends D, E, F and G lie in the same plane as B so that DEFG is a square.

(a) Calculate the length of each arm.

(b) (i) Calculate DB.

(ii) Part of the clothes line joins the points D, E, F and G. Calculate the total length of this part of the clothes line.



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Answers and hints ► page 125

Pythagoras and trigonometry in three dimensions (page 52)

1 (a) In ΔABC, $AC^2 = AB^2 + BC^2$

$AC^2 = 60^2 + 25^2 = 4225$

$AC = \sqrt{4225}\,\text{cm} = 65\,\text{cm}$

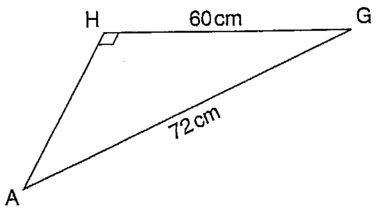
(b) In ΔACG, $AG^2 = AC^2 + CG^2$

$CG^2 = 72^2 - 65^2 = 959$

$CG = \sqrt{959} = 30{\cdot}96\ldots\text{cm}$

$= 31\,\text{cm}$ (to 2 s.f.)

(c) Angle required = HAG



$\sin HAG = \frac{60}{72}$

$\angle HAG = 56{\cdot}44\ldots^\circ$

$= 56^\circ$ (to nearest degree)

2 (a) In ΔABC, $AC = 47 \sin 35^\circ$ m

$= 26{\cdot}95\ldots\text{m} = 27\,\text{m}$ (to 2 s.f.)

(b) In ΔACT, $CT = AC \tan 18^\circ$

$= 8{\cdot}759\ldots^\circ\,\text{m} = 8{\cdot}8\,\text{m}$ (to 2 s.f.)

3 (a) In ΔMPT, $MT = 70 = PT \tan 32^\circ$

So $PT = \frac{70}{\tan 32^\circ} = 112{\cdot}02\ldots\text{m}$.

In ΔMTQ, $QT = \frac{70}{\tan 35^\circ} = 99{\cdot}97\ldots\text{m}$.

In ΔTPQ, using the cosine rule,

$PQ^2 = PT^2 + QT^2 - 2 \times PT \times QT \times \cos 134^\circ$

$= 38102{\cdot}3\ldots$

$PQ = 195{\cdot}19\ldots\text{m} \approx 195\,\text{m}$

(b) In ΔTPQ, using the sine rule,

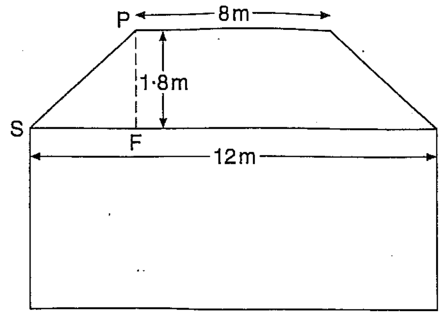
$$\frac{PQ}{\sin PTQ} = \frac{QT}{\sin TPQ}$$

$$\frac{195{\cdot}19\ldots}{\sin 134^\circ} = \frac{99{\cdot}97\ldots}{\sin TPQ}$$

$$\sin TPQ = \frac{99{\cdot}97\ldots \sin 134^\circ}{195{\cdot}19\ldots} = 0{\cdot}368\ldots$$

$\angle TPQ = 21{\cdot}61\ldots^\circ = 22^\circ$ (to nearest degree)

4 The sketch below shows a cross-section through the middle line of the roof:



F is the foot of the perpendicular from P onto the base plane of the roof, and $\angle PSF$ is the angle that PS makes with the horizontal.

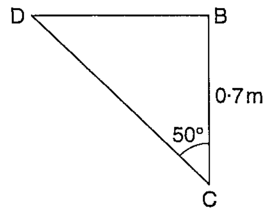
Since the house is symmetrical, we have

$SF = \frac{1}{2}(12 - 8)\text{m} = 2\text{m}$.

In ΔPFS, $\tan PSF = \frac{1{\cdot}8}{2} = 0{\cdot}9$ and

$\angle PSF = 41{\cdot}987\ldots^\circ = 42^\circ$ (to nearest degree)

5 (a)



In ΔCDB, $\frac{0{\cdot}7}{CD} = \cos 50^\circ$

$CD = \frac{0{\cdot}7}{\cos 50^\circ} = 1{\cdot}089\ldots\text{m}$

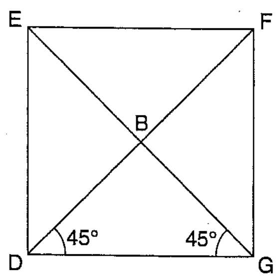
Length of arms = $1{\cdot}1$ m (to 2 s.f.)

(b) (i) In ΔCDB, $DB = 0{\cdot}7 \tan 50^\circ = 0{\cdot}8342\ldots\text{m}$

$= 0{\cdot}83\,\text{m}$ (to 2 s.f.)

Alternatively you could have used Pythagoras' rule.

(ii) Looking at the top view of the whole line, we see:



Since DEFG is a square, the angles in DBG will be 45°, with an angle of 90° at B.

We know that $DB = 0{\cdot}8342\ldots\text{m}$, and that $DB = BG$.

So $DG^2 = DB^2 + BG^2 = 2DB^2$

$= 2 \times (0{\cdot}8342\ldots)^2 = 1{\cdot}3918\ldots$

$DG = 1{\cdot}1797\ldots\text{m}$

Total length of clothes line $= 4 \times DG$

$= 4{\cdot}719\ldots\text{m} = 4{\cdot}72\,\text{m}$ (to 3 s.f.)

More help or practice

Using Pythagoras' rule in three dimensions
► Book Y3 pages 109 to 111

Angles in three dimensions ► Book Y4 pages 134 to 140