Topic 8: Exercises on the Ellipse
Level 2

1. For the ellipse $x^2 + 2y^2 = 4$, find (a) the eccentricity, (b) the coordinates of the foci, (c) the equations of the directrices. Sketch the ellipse.

(a) $\frac{1}{\sqrt{2}}$; (b) $(\pm\sqrt{2},0)$; (c) $x = \pm 2\sqrt{2}$

2. For the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ find (a) the eccentricity; (b) the coordinates of the foci; (c) the equations of the directrices. Sketch the ellipse.

(a) $\frac{3}{5}$; (b) $(0,\pm 3)$; (c) $y = \pm\frac{25}{3}$

3. A variable point $P(x, y)$ moves so that its distance from $(0,1)$ is one-half its distance from $y = 4$. Find the locus of P.

$$\frac{x^2}{3} + \frac{y^2}{4} = 1$$

4. Show that the equation $\frac{x^2}{29-k} + \frac{y^2}{4-k} = 1$, where k is a real number, represents an ellipse if $k < 4$. Show that the foci of the ellipse are independent of the value of k.

5. Find the parametric equations of the ellipse $x^2 + 4y^2 = 4$.

(a) $x = 2\cos\theta$, $y = \sin\theta$; (b) $x = 2\sec\theta$, $y = 2\tan\theta$

6. Find the Cartesian equations of the ellipse $x = 5\cos\theta$, $y = 4\sin\theta$.

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

7. The points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. If PQ subtends a right angle at $(a, 0)$. Show that $\tan\frac{\theta}{2}\tan\frac{\phi}{2}=-\frac{b^2}{a^2}$.

8. The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci $S(al, 0)$ and $S'(-al, 0)$.
Show that (a) $PS = a(1-e\cos\theta)$ and $PS' = a(1+e\cos\theta)$; (b) $PS + PS' = 2a$.

9. Find the equations of the tangent and the normal to the ellipse $3x^2 + 4y^2 = 48$ at the point $(2,-3)$.

$x - 2y = 8,\ 2x + y = 1$

10. Find the equations of the tangent and the normal to the ellipse $x = 4\cos\theta, y = 2\sin\theta$ at the point where $\theta = -\frac{\pi}{4}$.

$x - 2y = 4\sqrt{2},\ 2x + y = 3\sqrt{2}$

11. Find the equation of the chord of contact of tangents to the ellipse $3x^2 + 4y^2 = 48$ from the point $(6,4)$.

$9x + 8y = 24$

12. The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The tangent at P meets the tangents at the ends of the major axis at Q and R. Show that QR subtends a right angle at either focus. Deduce that if P is the point $\left(1, \frac{2\sqrt{2}}{3}\right)$ lies on the ellipse $\frac{x^2}{9} + y^2 = 1$ with foci S and S', then Q, S, R, S' are concyclic, and find the equation of the circle through these points.

$$x^2 + \left(y - \frac{3}{2\sqrt{2}}\right)^2 = \frac{73}{8}$$