Topic 9: Exercises on the Hyperbola
Level 3

1. A variable point $P(x,y)$ moves so that its distance from $(0,4)$ is two times its distance from $y=1$. Find the locus of P.

$$\boxed{\frac{y^2}{4}-\frac{x^2}{12}=1}$$

2. The asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are inclined to each other at an angle α.

Show that $\tan\alpha=\frac{2ab}{\left|a^2-b^2\right|}$.

3. A point $P(a \sec \theta, b \tan \theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with foci $S(a\,l, 0)$ and $S'(-a\,l, 0)$. (a) Show that $PS = a|e \sec \theta - 1|$ and $PS' = a|e \sec \theta + 1|$. (b) Deduce that $|PS - PS'| = 2a$.

4. Points $P(a \sec \theta, b \tan \theta)$ and $Q(a \sec \phi, b \tan \phi)$ lie on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. (a) Use the result that the chord PQ has equation $\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$ to show that if PQ is a local chord, then $\tan\frac{\theta}{2}\tan\frac{\phi}{2}$ takes one of the values $\frac{1-e}{1+e}$ or $\frac{1+e}{1-e}$. (b) The point $P\left(2\sqrt{3}, 3\sqrt{3}\right)$ is one extremity of a focus chord on the hyperbola $\frac{x^2}{3} - \frac{y^2}{9} = 1$. Find the coordinates of the other extremity Q.

$\left(2\sqrt{3}, -3\sqrt{3}\right)$ or $\left(-\frac{14\sqrt{3}}{13}, \frac{9\sqrt{3}}{13}\right)$

5. A point $P(a \sec\theta, b \tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The line through P perpendicular to the x-axis meets an asymptote at Q and the normal at P meets the x-axis at N. Show that ON is perpendicular to the asymptote.

6. The point $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is joined to the vertices $A(a,0)$ and $A'(-a,0)$. The lines AP and $A'P$ meet he asymptote $y=\frac{b}{a}x$ at Q and R respectively. (i) Find the coordinates of Q and R. (ii) Hence find the length QR, showing that it is independent of θ, and show that the area of triangle PQR is $\frac{1}{2}\left|ab(\sec\theta-\tan\theta)\right|$ square units.

(i) $\left(\dfrac{a\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}, \dfrac{b\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right), \left(\dfrac{a\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}, \dfrac{b\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right)$; (ii) $\sqrt{a^2+b^2}$

7. Find the equation of the tangent and normal to (a) the hyperbola $\frac{x^2}{12} - \frac{y^2}{27} = 1$ at the point $(4,3)$; (b) the hyperbola $x = 3\sec\theta, y = 6\tan\theta$ at the point where $\theta = \frac{\pi}{6}$.

(a) $3x - y = 9$, $x + 3y = 13$; (b) $4x - y = 6\sqrt{3}$, $x + 4y = 10\sqrt{3}$

8. The point $P(a \sec\theta, b \tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The points T and T' are the feet of the perpendiculars from the foci S and S' respectively to this tangent. Show that $ST \cdot S'T' = b^2$.

9. Show that if $y = mx + k$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $m^2a^2 - b^2 = k^2$.

Hence find the equation of the tangents from the point $(1,3)$ to the hyperbola $\frac{x^2}{4} - \frac{y^2}{15} = 1$ and the coordinates of their points of contact.

$y = 2x + 1$, $(-8,-15)$; $y = -4x + 7$, $\left(\frac{16}{7}, -\frac{15}{7}\right)$