Topic 10: Exercises on the Rectangular Hyperbola
Level 3

1. A point $P(a\sec\theta, a\tan\theta)$ lies on the rectangular hyperbola $x^2 - y^2 = a^2$. A is the point $(a,0)$. M is the midpoint of AP. Find the equation of the locus of M.

$$(2x-a)^2 - (2y)^2 = a^2$$

2. For the rectangular hyperbola $xy = 18$, find (a) the eccentricity; (b) the coordinates of the foci; (c) the equations of the directrices, (d) the equations of the asymptotes. Sketch the rectangular hyperbola.

(a) $\sqrt{2}$; (b) $(6,6), (-6,-6)$; (c) $x + y = \pm 6$; (d) $x = 0, y = 0$

3. Show that if $y = mx + k$ is a tangent to the rectangular hyperbola $xy = c^2$, then $k^2 + 4mc^2 = 0$. Hence find the equation of the tangents from the point $(-1,-3)$ to the rectangular hyperbola $xy = 4$ and find the coordinates of their points of contact.

$y = -x - 4, (-2,-2);\ y = -9x - 12, \left(-\frac{2}{3},-6\right)$

4. The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ lie on the rectangular hyperbola $xy = c^2$. The tangents at P and Q meet at R, and OR cuts PQ at M. Show that M is the midpoint of PQ.

5. The point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. The normal at P meets the rectangular hyperbola $x^2 - y^2 = a^2$ at Q and R. Show that P is the midpoint of QR.

6. The point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. The normal at P meets the hyperbola again at Q. The circle on PQ as diameter meets the hyperbola again at R. Find the coordinates of Q and R.

$$Q\left(-\frac{c}{t^3}, -ct^3\right),\ R\left(-ct, -\frac{c}{t}\right)$$

7. The point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. The normal at P meets the hyperbola again at Q. M is the midpoint of PQ. Find the equation of the locus of M.

$$4x^3y^3 + c^2(x^2 - y^2)^2 = 0$$

8. The point $P\left(ct, \frac{c}{t}\right)$, where $t \neq 1, t \neq -1$, lies on the rectangular hyperbola $xy = c^2$. The tangent at P meets the x-axis and the y-axis at Q and R respectively. The normal at P meets the lines $y = x$ and $y = -x$ at S and T respectively. Show that $QSRT$ is a rhombus.

9. The point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. Show that the normal at P cuts the hyperbola again at the point Q with coordinates $\left(-\frac{c}{t^3}, -ct^3\right)$. Hence find the coordinates of the point R where the normal at Q cuts the hyperbola again.

$\left(ct^9, \frac{c}{t^9}\right)$

10. The point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. The normal at P meets the x-axis at A and the tangent at P meets the y-axis at B. M is the midpoint of AB. Find the equation of the locus of M as P moves on the hyperbola.

$$\boxed{2c^2xy = c^4 - y^4}$$