Topic 19: Exercises on Harder 3 Unit Inequalities
Level 2, Part 1

1. Show that $\dfrac{a+b}{2} \le \sqrt{\dfrac{a^2+b^2}{2}}$.

2. If $a>0$ and $b>0$, show that $\sqrt{ab} \ge \left(\dfrac{\frac{1}{a}+\frac{1}{b}}{2}\right)^{-1}$.

3. Show that $(a^2 - b^2)(c^2 - d^2) \le (ac - bd)^2$.

4. If $a > 0$ and $b > 0$, show that $\frac{1}{a} + \frac{1}{b} \ge \frac{4}{a+b}$.

5. If $a > b$ and $b > 0$, show that $\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{8}{(a+b)^2}$.

6. Show that $(ac+bd)^2 \le (a^2+b^2)(c^2+d^2)$.

7. Show that $(lx+my+nz)^2 \le (l^2+m^2+n^2)(x^2+y^2+z^2)$.

8. Show that $xy + yz + zx \le x^2 + y^2 + z^2$.

9. Show that $(x + y + z)^2 \le 3(x^2 + y^2 + z^2)$.

10. Show that $(a^3 + b^3 + c^3)^2 \le (a^2 + b^2 + c^2)(a^4 + b^4 + c^4)$.

Hint: Use the inequality $(lx + my + nz)^2 \le (l^2 + m^2 + n^2)(x^2 + y^2 + z^2)$.

11. Show that $a^4 + b^4 + c^4 \ge a^2b^2 + b^2c^2 + c^2a^2 \ge abc(a + b + c)$.

12. If $a > 0$, $b > 0$ and $c > 0$, show that $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$.

Hint: Inspect that $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$.