Topic 19: Exercises on Harder 3 Unit Inequalities
Level 3, Part 1

1. If $a > 0$, show that $a^2 + \dfrac{1}{a^2} \geq a + \dfrac{1}{a} \geq 2.$

2. If $a \geq 0, b \geq 0$ and $c \geq 0$, show that $ab + bc + ca \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}.$

3. If $a_1 \geq 0, a_2 \geq 0, b_1 \geq 0$ and $b_2 \geq 0$, show that $\sqrt{(a_1 + b_1)(a_2 + b_2)} \geq \sqrt{a_1 a_2} + \sqrt{b_1 b_2}$.

4. Show that if a, b and c are the sides of a triangle, then $(a + b - c)(b + c - a)(c + a - b) \leq abc$.
Hint: Show that if $x \geq 0$, $y \geq 0$ and $z \geq 0$, then $(x + y)(y + z)(z + x) \geq 8xyz$.

5. If $a_1, a_2, \cdots, a_n$ are positive numbers such that $a_1 a_2 \cdot \ldots \cdot a_n = 1$, show that $(1+a_1)(1+a_2)\cdot \ldots \cdot (1+a_n) \geq 2^n$.

6. If $a > 0, b > 0$ and $c > 0$, show that $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$.

7. If $a>0, b>0, c>0$ and $d>0$, show that

$$\frac{16}{a+b+c+d} \le \frac{3}{b+c+d} + \frac{3}{a+b+c} + \frac{3}{a+b+d} + \frac{3}{a+c+d} \le \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$$

Hint: Use the inequality $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \ge 9$ with positive x, y and z.

8. Show that $\dfrac{a^2+b^2+c^2}{3} \geq \left(\dfrac{a+b+c}{3}\right)^2$.

9. If $x \neq 0$ and $y \neq 0$, show that $x^4 + y^4 \leq \frac{x^6}{y^2} + \frac{y^6}{x^2}$.

10. If $a > 1, b > 1$ and $c > 1$ such that $\frac{a}{b} \geq \frac{c}{a}$, show that $\frac{\lg a}{\lg b} \geq \frac{\lg c}{\lg a}$.

11. If $x > y$ and $xy = 1$, show that $\frac{x^2 + y^2}{x - y} \geq 2\sqrt{2}$.