Topic 19: Exercises on Harder 3 Unit Inequalities
Level 3, Part 2

1. Show that the geometric mean of n positive numbers can not exceed their arithmetic mean, i.e., if $x_1, x_2, ..., x_n$ are positive numbers, show that

$$(x_1 x_2 ... x_n)^{1/n} \leq \frac{x_1 + x_2 + ... + x_n}{n}.$$

Hint: Show that $x \leq e^{x-1}$ for all x.

2. If $a_1, a_2, ..., a_n$ are positive numbers, show that

$$(a_1 + a_2 + ... + a_n) \cdot \left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n} \right) \geq n^2.$$

3. If $a > 0, b > 0$, $c > 0$ and $a + b + c = 1$, show that $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 27$.

4. If $a_1, a_2, ..., a_n$ are positive numbers, show that

(i) $a_1^n + a_2^n + ... + a_n^n \geq n a_1 a_2 ... a_n$

(ii) $\sqrt[n]{a_1 a_2 \ldots a_n} \geq \dfrac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \ldots + \dfrac{1}{a_n}}$.

5. Show that $x - \frac{1}{3}x^3 < \tan^{-1} x < x - \frac{1}{3}x^3 + \frac{1}{5}x^5$ for $x > 0$.

6. Show that $xy \le e^{x-1} + y \ln y$ for all real x and all positive y. When does equality hold?

7. Show that $\frac{1}{2}\int_0^1 x^4(1-x)^4\,dx < \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx < \int_0^1 x^4(1-x)^4\,dx$ and hence deduce that $\frac{22}{7}-\frac{1}{630} < \pi < \frac{22}{7}-\frac{1}{1260}$.

8. Show that $1-e^{-\pi/2} < \int_0^{\pi/2} e^{-\sin x}\,dx < \frac{\pi}{2e}(e-1)$. Hint: if $\frac{\pi}{2} > x > 0$, show that $x > \sin x > \frac{2}{\pi}x$.

9. If $b > a > 0$, show that $\dfrac{\ln b - \ln a}{b - a} < \dfrac{1}{2}\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$.

10. If $b > a > 0$, show that $\dfrac{b-a}{\ln b - \ln a} < \dfrac{a+b}{2}$.