Topic 5: Exercises on Polynomials II
Level 2

1. Find the roots of $P(x)=0$ over (i) the rational numbers, (ii) the real numbers, (iii) the complex numbers, if

(a) $P(x)=x^4-5x^2+6$

i) no roots, ii, iii) $\pm\sqrt{2},\pm\sqrt{3}$

(b) $P(x)=x^4-x^2-2$;

i) no roots, ii) $\pm\sqrt{2}$, iii) $\pm\sqrt{2},\pm i$

(c) $P(x)=x^4+5x^4+4$.

i, ii) roots, iii) $\pm i,\pm 2i$

2. Divide $P(x) = x^3 - 3x^2 + 4x - 2$ by $x + 1$ and verify that the remainder is $P(-1)$.

$$P(x) = (x+1)(x^2 - 4x + 8) - 10\,;\, P(-1) = -10$$

3. Divide $P(x) = x^3 - x^2 + x - 1$ by $x - i$ and verify that the remainder is $P(i)$.

$$P(x) = (x-i)\{x^2 + (i-1)\,x - i\}\,;\; P(i) = 0$$

4. Express $P(x) = x^3 - 2x^2 + 4x - 8$ as a product of irreducible factors over (i) **Q**, (ii) **R**, (iii) **C**.

(i, ii) $P(x) = (x-2)(x^2+4)$, (iii) $P(x) = (x-2)(x-2i)(x+2i)$

5. Express $P(x) = x^4 - x^3 - 5x^2 - x - 6$ as a product of irreducible factors over (i) **Q**, (ii) **R**, (iii) **C**.

(i, ii) $P(x) = (x+2)(x-3)(x^2+1)$; (iii) $P(x) = (x+2)(x+3)(x-i)(x+i)$

6. Find $P(x)$, given that $P(x)$ is monic, of degree 4, with -1 as a single zero and 3 as a zero of multiplicity 3.

$$P(x) = x^4 - 8x^3 + 18x^2 - 27$$

7. If $P(x) = 4x^3 + 15x^2 + 12x - 4$ has a double zero, find all the zeros and factorise $P(x)$ fully over the real numbers.

$$P(x) = (x+2)^2(4x-1)$$

8. If $P(x) = x^3 - 6x^2 + 9x + c$ for some real number c, find the values of x for which $P(x)' = 0$. Hence find the values of c for which the equation $P(x) = 0$ has a repeated root.

$x = 1,\ x = 3;\ c = -4,\ c = 0.$

9. If $P(x) = x^4 - 3x^3 - 6x^2 + 28x - 24$ has a triple zero, find all the zeros and factorise $P(x)$ over the real numbers.

$P(x) = (x-2)^3(x+3)$

10. If $P(x) = x^4 + 2x^3 - 12x^2 - 40x + c$ has a triple zero, find c and factorise $P(x)$ over the real numbers.

$c = -32$; $P(x) = (x+2)^3(x-4)$

11. If $P(x) = 1 - x - \frac{x^2}{2!} - \cdots + (-1)^n \frac{x^n}{n!}$, show that $P(x)$ has no multiple zero for $n \geq 2$.

12. Given that $P(x)$ has a rational zero, find this zero and factorise $P(x)$ over the real numbers if (a) $P(x)=2x^3-3x^2+2x-3$, (b) $P(x)=2x^3+x^2-4x-2$.

(a) $x=\frac{3}{2}$; $P(x)=(2x-3)(x^2+1)$; (b) $x=-\frac{1}{2}$; $P(x)=(2x+1)(x-\sqrt{2})(x+\sqrt{2})$

13. If $P(x) = 3x^4 - 4x^3 - 14x^2 - 4x + 3$, solve the equation $P(x) = 0$ over **C** and factor $P(x)$ fully over **R**.

$x = -1, -1, 1/3, 3;\ P(x) = (3x-1)(x-3)(x+1)^2$