Topic 5: Exercises on Polynomials II
Level 3

1. Express $P(x)=(x-a)\{2Q(x)+(x-a)Q'(x)\}+c$ as a product of irreducible factors (a) over **Q**; (b) over **R**; (c) over **C**.

(a), (b) $P(x)=(x-a)^2Q(x)+cx+d$; (c) $P(x)=(x-1)(x+5)(x-\sqrt{2}\,i)(x+\sqrt{2}\,i)$

2. $P(x)$ is a monic polynomial of degree 4 with integer coefficients and constant term 4. One zero is $\sqrt{2}$, another zero is rational and the sum of the zeros is positive. Factorise $P(x)$ fully over **R**.

$P(x)=(x-\sqrt{2})(x+\sqrt{2})(x+1)(x-2)$

3. The equation $z^n - 1 = 0$ has roots $1, z_1, z_2, \ldots, z_{n-1}$. Show that $(1-z_1)(1-z_2)\ldots(1-z_{n-1}) = n$.

4. If $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$, show that $P(x) = 0$ has a multiple root and find this root and its multiplicity.

-1 is a root of multiplicity 3

5. If $ax^3 + bx^2 + d = 0$ has a double root, show that $27a^2d + 4b^3 = 0$.

6. Find the integers m and n such that $(x+1)^2$ is a factor of the polynomial $P(x) = x^5 + 2x^2 + mx + n$.

$m = -1;\ n = -2$

7. The equation $x^3+3px^2+3qx+r=0$, where $p^2 \neq q$, has a double root.
Show that $4(p^2-q)(q^2-pr)=(pq-r)^2$.

8. The equation $x^n + px - q = 0$ has a double root. Show that $\left(\frac{p}{n}\right)^n + \left(\frac{q}{n-1}\right)^{n-1} = 0$.

9. Show that if the polynomials $P_1(z) = b_n z^n + b_{n-1} z^{n-1} + \ldots + b_0$ and $P_2(z) = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_0$ are equal for more than n values of z, then $b_n = c_n, b_{n-1} = c_{n-1}, \ldots, b_0 = c_0$.

10. Find the remainder when $P(x) = x^3 + 2x^2 - 1$ is divided by (a) $x - i$; (b) $x^2 + 1$.

(a) $-3 - i$; (b) $-x - 3$.

11. When $P(x) = x^4 + ax^2 + bx$ is divided by $x^2 + 1$, the remainder is $x + 2$. Find the values of a and b.

$a = -1, b = 1$.

12. Find the values of the real numbers p and q if x^2+1 is a factor of the polynomial $P(x)=x^4+px^3+2x+q$. Hence factorise $P(x)$ over **R** and over **C**.

$p=2$, $q=-1$; over **R** $P(x)=(x^2+1)(x+1-\sqrt{2})(x+1+\sqrt{2})$; over **C** $P(x)=(x-i)(x+i)(x+1-\sqrt{2})(x+1+\sqrt{2})$

13. (i) Show that the remainder when the polynomial $P(x)$ is divided by $(x-a)^2$ is $P'(a)x + P(a) - a\,P'(a)$.

(ii) Find the value of *k* for which $(x-1)$ is a factor of the polynomial $P(x) = x^{11} - 3x^6 + kx^4 + x^2$.
For this value of *k*, find the remainder on dividing $P(x)$ by $(x-1)^2$.

(ii) $k = 1$; $(1-x)$.