KAMBALA

EXTENSION 2 MATHEMATICS

HSC ASSESSMENT TASK #1

NOVEMBER 2004

Time Allowed: 50 minutes

INSTRUCTIONS

- This task contains 3 questions of 12 marks each. Marks for each part question are shown.
- Answer all questions on the writing paper provided.
- Start each question on a new page.
- Calculators may be used.
- Show all necessary working.
- Marks may be deducted for careless or badly arranged work.

Question 1 **Start a new page** **Marks**

(a) Simplify, leaving your answer in the form $a+ib$:

(i) $(2+3i)(1-2i)$ 1

(ii) $\dfrac{\sqrt{3}-i}{\sqrt{2}-i}$ 2

(b) (i) Write $1+i\sqrt{3}$ in mod-arg form 2

(ii) Hence find the two square roots of $1+i\sqrt{3}$ 3

(d) If $z_1=4i$, $z_2=2\sqrt{3}-2i$ and $z_3=-2\sqrt{3}-2i$ represent A, B and C respectively, show that $\Delta\,ABC$ is equilateral 4

Question 2 **Start a new page** **Marks**

(a) (i) Find $(1-i)^5$ in mod-arg form 2

(ii) Using De Moivres Theorem or otherwise find the values of n for which $(1-i)^n$ is purely real. 2

(b) For what values of a and b is $z=1$ a solution to the equation

$$2iz^2+(ia+1)z-(i+3b)=0\,?$$ 3

(c) (i) Solve the equation $z^5 - i = 0$, giving the roots z_1, z_2, z_3, z_4, z_5 in mod-arg form with principal arguments. **3**

(iii) Show that the perimeter of the pentagon $z_1z_2z_3z_4z_5$ formed by the roots in the Argand plane is given by **2**

$$P = 5\sqrt{2\left(1-\cos\frac{2\pi}{5}\right)}$$

Question 3 **Start a new page** **Marks**

(a) Sketch the region on the Argand plane where both **3**

$1 \le |z - i| \le 2$ and $\frac{\pi}{3} < \arg z < \frac{2\pi}{3}$ apply

(b) By looking at the values of i^n for $n = 1, 2, 3, 4, \ldots$ or otherwise, find the possible values of the series $S_n = 1 + i + i^2 + i^3 + i^4 + \ldots\ldots + i^n$ **3**

(c) z is the complex number $z = x + iy$.

(i) Find the equation of the curve for which $\text{Im}(z^2) = 4$ and sketch on a suitably labelled Argand plane **2**

(ii) Show that the locus given by $\text{Re}(z^2) = 0$ is the pair of intersecting lines $y = \pm x$ **2**

(iii) The region R in the Argand plane consists of all points which satisfy both $0 \le \text{Im}(z^2) \le 4$ and $\text{Re}(z^2) \le 0$. Complete a sketch of the region R on your answer for part (i) **2**

End of task

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \ne -1; \quad x \ne 0, \text{ if } n < 0$

$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \ne 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \ne 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \ne 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \ne 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \ne 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \ne 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \; -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE: $\ln x = \log_e x, \quad x > 0$

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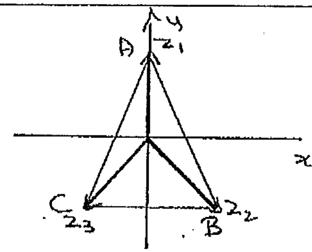
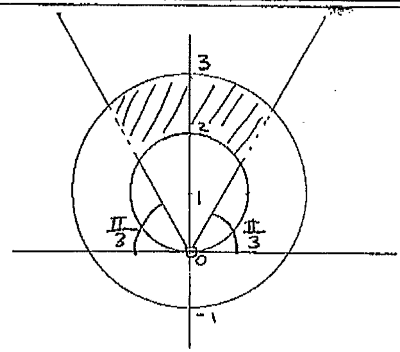
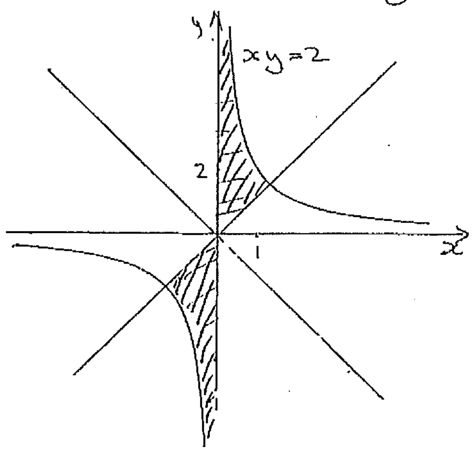
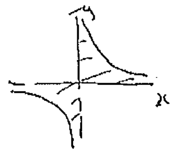
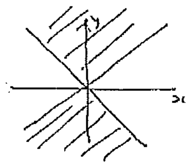
Qn	Solutions	Marks	Comments+Criteria
1(a)	(i) $(2+3i)(1-2i) = 2-4i+3i-6i^2$		
	$= 8-i$	✓	
	(ii) $\frac{\sqrt{3}-i}{\sqrt{2}-i}\cdot\frac{\sqrt{2}+i}{\sqrt{2}+i} = \frac{\sqrt{3}+i\sqrt{3}-i\sqrt{2}-i^2}{3}$	✓	
	$= \frac{(1+\sqrt{6})}{3} + i\frac{(\sqrt{3}-\sqrt{2})}{3}$	✓	
(b)	(i) $1+i\sqrt{3} = z$ $\quad \lvert z\rvert = \sqrt{1^2+(\sqrt{3})^2} = 2$		
	$\arg z = \tan^{-1}\sqrt{3}$ in Q1 $= \frac{\pi}{3}$		
	$\therefore z = 2\,\text{cis}\frac{\pi}{3}$	✓✓	
	(ii) $z = 1+i\sqrt{3}$		
	$z = 2\text{cis}\left(\frac{\pi}{3}+2k\pi\right)$ $\quad k=0,\pm1,\dots$		
	$w = z^{\frac{1}{2}} = 2^{\frac{1}{2}}\,\text{cis}\,\frac{1}{2}\left(\frac{\pi}{3}+2k\pi\right)$	✓	
	$= \sqrt{2}\,\text{cis}\left[\frac{1}{2}\left(\frac{\pi}{3}+2k\pi\right)\right]$		
	$\therefore r = \sqrt{2}, \quad \theta_1 = \frac{\pi}{6}$	✓	
	$\theta_2 = \frac{\pi}{6}+\pi = \frac{7\pi}{6} = -\frac{5\pi}{6}$		
	$\therefore w_1 = \sqrt{2}\,\text{cis}\frac{\pi}{6} \quad w_2 = \sqrt{2}\,\text{cis}-\frac{5\pi}{6}$	✓	

Qn	Solutions	Marks	Comments+Criteria
(c)		✓	
	now $\arg(z_1-z_2) = \arg(-2\sqrt{3}+2i)$ $= \tan^{-1}\frac{1}{\sqrt{3}}$ in Q2 $= \frac{2\pi}{3}$		or similar proof
	$\therefore A\hat{B}C = \frac{\pi}{3}$	✓	
	Similarly $A\hat{C}B = \frac{\pi}{3}$ using $\arg(z_1-z_3)$	✓✓	
	$\therefore \triangle ABC$ equilateral. by $\angle$ sum		
2(a)	(i) $z^5 = (1-i)^5 \quad z = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$	✓	1 for correct mod arg form
	$z^5 = (\sqrt{2})^5\,\text{cis}\left(-\frac{\pi}{4}\right)$		
	$= 4\sqrt{2}\,\text{cis}-\frac{5\pi}{4}$		
	$= 4\sqrt{2}\,\text{cis}\frac{3\pi}{4}$	✓	$-\frac{1}{2}$ for non principal argument
	(ii) $z^n = (\sqrt{2})^n\,\text{cis}\,n\left(-\frac{\pi}{4}\right)$		
	$= 2^{\frac{n}{2}}\,\text{cis}\left(-\frac{n\pi}{4}\right)$	✓	
	for $n = 0, \pm4, \pm8, \pm12 \dots$	✓	

Qn	Solutions	Marks	Comments+Criteria
2(b)	$2iz^2+(ia+1)z-(i+3b)=0$ let $z=1$ $\therefore 2i+ia+1-i-3b=0$ $1-3b+i(2+a-1)=0$ $1-3b+i(1+a)=0$ $\therefore b=\frac{1}{3}\quad a=-1$ by equating coefficients	✓ ✓ ✓	
2(c)	(i) $z^5-i=0$ let $z=r\,\text{cis}\,\theta$ $\therefore r^5\,\text{cis}\,5\theta=\text{cis}\left(\frac{\pi}{2}+2k\pi\right)$ $k=0,\pm1,\dots$ $\therefore r=1\quad 5\theta=\frac{\pi}{2}+2k\pi$ $\theta=\frac{\pi}{10}+\frac{2k\pi}{5}\quad k=0,\pm1,\pm2,\dots$ $\therefore z_1=\text{cis}\frac{\pi}{10}$ $\quad z_4=\text{cis}\frac{13\pi}{10}=\text{cis}\frac{-7\pi}{10}$ $z_2=\text{cis}\frac{\pi}{2}$ $z_3=\text{cis}\frac{9\pi}{10}$ $\quad z_5=\text{cis}\frac{17\pi}{10}=\text{cis}\frac{-3\pi}{10}$ (ii) $\measuredangle$ between each root is $\frac{2\pi}{5}$ and pentagon is regular. Perimeter is $5\times$ length of each side $\therefore$ length of each side by cosine rule	✓ ✓ ✓ ✓	accept non-principal arguments but $\left(-\frac{1}{2}\right)$

Qn	Solutions	Marks	Comments+Criteria
	ie $a^2=1^2+1^2-2\cdot1\cdot1\cdot\cos\frac{2\pi}{5}$ $=2-2\cos\frac{2\pi}{5}$ $=2\left(1-\cos\frac{2\pi}{5}\right)$ $\therefore$ perimeter of pentagon is $P=5\sqrt{2\left(1-\cos\frac{2\pi}{5}\right)}$	✓	
3(a)	 	✓✓✓	1 circle loci 1 arg loci 1 region shading
(b)	now $S_n=\frac{a(r^n-1)}{r-1}$ $\quad a=1,\ r=i$ $\therefore S_n=\frac{i^n-1}{i-1}$ $n=1\quad S_1=\frac{i-1}{i-1}=1$ $S_2=\frac{i^2-1}{i-1}=i+1$ $S_3=\frac{i^3-1}{i-1}=i^2+i+1=i$ $S_4=\frac{i^4-1}{i-1}=\frac{0}{i-1}=0$ $S_5=\frac{i^5-1}{i-1}=1=S_1$ etc	✓	

Qn	Solutions	Marks	Comments+Criteria
	$\therefore S_n = 1,\ i+1,\ i,\ 0$ $S_n = 1$ for $n = 1, 5, 9, \dots$ $= 4k-3 \quad k = 1, 2, 3,$ $S_n = i+1$ for $n = 2, 6, 10, \dots$ $= 4k-2$ $S_n = i$ for $n = 3, 7, 11, \dots$ $= 4k-1$ $S_n = 0$ for $n = 4, 8, 12 \dots$ $= 4k$	✓ ✓	
(c)	(i) $\mathrm{Im}(z^2) = 4$ ie $z = x + iy$ $z^2 = x^2 - y^2 + 2xyi$ $\mathrm{Im}(z^2) = 4$ is $2xy = 4$ $xy = 2$ 	✓ ✓	

Qn	Solutions	Marks	Comments+Criteria
3(c)(ii)	$\mathrm{Re}(z^2) = 0$ is $x^2 - y^2 = 0$ $y^2 = x^2$ $y = \pm\|x\|$ $= \pm x$	✓ ✓	
(iii)	$0 \leq \mathrm{Im}(z^2) \leq 4$ is region  $0 \leq 2xy \leq 4$ $0 \leq xy \leq 2$ $\mathrm{Re}(z^2) \leq 0$ is $x^2 - y^2 \leq 0$ $x^2 \leq y^2$ 		2 for shading correctly