Neap:

HSC Trial Examination 2012

Mathematics Extension 1

This paper must be kept under strict security and may only be used on or after the morning of Thursday 9 August, 2012 as specified in the Neap Examination Timetable.

General Instructions

Reading time – 5 minutes

Working time – 2 hours

Write using black or blue pen

Board-approved calculators may be used

A table of standard integrals is provided at the back of this paper

All necessary working should be shown in every question

Section I – 10 marks

10 multiple-choice questions

Section II – 60 marks

4 short-answer questions

Total marks – 70

Attempt questions 1–14

Section I – 10 marks
Attempt Questions 1–10
All questions are of equal value

Use the multiple-choice answer sheet for Questions 1–10.

1. Which of the following is the solution of $\frac{x^2}{x^2-9} > 0$?
   (A) $x > 3$
   (B) $x < -3$
   (C) $-3 < x < 3$
   (D) $x < -3, x > 3$

2. What is the exact value of $\operatorname{cosec}\left(-\frac{7\pi}{4}\right)$?
   (A) $\frac{1}{\sqrt{2}}$
   (B) $-\frac{1}{\sqrt{2}}$
   (C) $\sqrt{2}$
   (D) $-\sqrt{2}$

3. What is the domain and range of $y = 2\sin^{-1}\left(\frac{x}{3}\right)$?
   (A) $x \le |3|, y \le |\pi|$
   (B) $-1 \le x \le 1, -3 \le y \le 3$
   (C) $-1 \le x \le 1, -\pi \le y \le \pi$
   (D) $-3 \le x \le 3, -2 \le y \le 2$

4. Which of the following functions is the inverse function of $f(x) = 3 - \frac{1}{2x+6}$?
   (A) $f^{-1}(x) = 6 - \frac{2}{x+4}$
   (B) $f^{-1}(x) = 6 - \frac{1}{6-2x}$
   (C) $f^{-1}(x) = -3 - \frac{1}{2x-6}$
   (D) $f^{-1}(x) = 3 - \frac{1}{6-2x}$

5. Using a suitable substitution, $\int_0^{\frac{\pi}{2}} (\sin x \cdot \cos^4 x)dx$ can be written as

(A) $-\int_0^1 u^4 du$

(B) $\int_0^1 u^4 du$

(C) $-\int_0^{\frac{\pi}{2}} u^4 du$

(D) $\int_0^1 u(1-u^2)^{\frac{1}{2}} du$

6. If $y = \sin^{-1}\left(\frac{a}{x}\right)$, then $\frac{dy}{dx}$ equals

(A) $\frac{-a}{x^2\sqrt{x^2-a^2}}$

(B) $\frac{x}{\sqrt{x^2-a^2}}$

(C) $\frac{-x}{\sqrt{x^2-a^2}}$

(D) $\frac{-a}{x\sqrt{x^2-a^2}}$

7. A solution to the integral $\int \frac{1}{c(a-bx)}dx$, given that a, b and c are constants, could be

(A) $-\frac{1}{bc}\log_e(a-bx)$

(B) $\frac{1}{c}\log_e(a-bx)$

(C) $\log_e(a-bx)^{bc}$

(D) $\log_e c(a-bx)$

8. If $\sin x = \frac{3}{5}, \frac{\pi}{2} \le x \le \pi$, then $\tan 2x$ would be equal to

(A) $-\frac{12}{7}$

(B) $\frac{12}{7}$

(C) $-\frac{24}{7}$

(D) $\frac{24}{7}$

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9. Given $f(x) = -x^2 + 2x + 3$, the graph of $y = \frac{1}{f(x)}$ has

(A) asymptotes at $x = 1$ and $x = -3$.

(B) x-intercepts at $x = -1$ and $x = 3$.

(C) asymptotes at $x = -1$ and $x = 3$.

(D) an x-intercept at $x = -\frac{1}{3}$.

10. What ratio does the point $P(10, 11)$ divide the interval AB, where $A(-2, 3)$ and $B(7, 9)$?

(A) 1 : 4

(B) 4 : –1

(C) 1 : –4

(D) 4 : 1

Marks

Section II – 60 marks
Attempt Questions 11–14
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) There are five women and six men in a group. From this group, a committee of four is to be chosen.

(i) How many different ways can a committee be formed that contain three women and one man? **2**

(ii) A particular man and a particular woman are to be on the committee. **1**

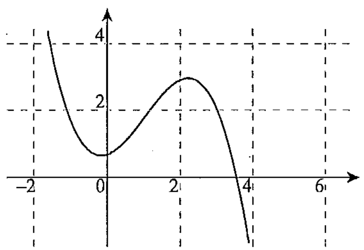
What is the probability that this committee is formed?

(b) The function $f(x) = x^2 - e^{(x-1)} + 1$ has one root between $x = 3$ and $x = 4$.

(i) Show that the root lies between $x = 3$ and $x = 4$. **1**

(ii) Hence, find a better approximation for the root using $x_0 = 3.5$ with one application of Newton's Method. **2**

(c) **4**



The polynomial $P(x) = ax^3 + bx^2 - 6x + 2$ has a factor of $(x - 1)$ and leaves a remainder of 6 when divided by $(x + 2)$.

Find the values of a and b and hence express $P(x)$ as a product of linear factors.

(d) (i) Show that $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$. **2**

(ii) Hence, if α, β and γ are the angles of ΔABC and $\sin\gamma = 2\sin\alpha\cos\beta$, prove that ΔABC is an isosceles triangle. **3**

Marks

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Given that $y = e^{2x} + e^{-2x}$, determine the values of constants a and b that satisfy the following equation: **4**

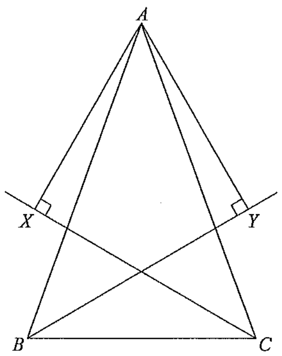
$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 5e^{2x} + e^{-2x}$$

(b) Find $\int_0^{\pi} \frac{4dx}{\sqrt{16 - x^2}}$. **3**

(c) AB and AC are equal chords of a circle.

Prove that the tangent at A is parallel to BC. (Drawing a diagram to represent this information may be helpful.) **3**

(d) Prove that the vertex of an isosceles triangle is equidistant from the bisectors of the base angles. **5**



Marks

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) $P(2at, at^2)$ is a variable point on the parabola $x^2 = 4ay$, where the focus is S.

Q divides the interval from P to S in the ratio $t^2 : 1$, where $x = \dfrac{2at}{t^2+1}$ and $y = \dfrac{2at^2}{t^2+1}$.

(i) Show that $\dfrac{y}{x} = t$. **1**

(ii) Prove that as P moves, Q moves in a circle, and state the centre of the circle. **2**

(b) Find the coefficient of x^5 in the expansion of $\left(2x - \dfrac{1}{x}\right)^{11}$. **3**

(c) A multiple-choice test contains ten questions. Each question has four choices for the correct answer. Only one of the choices is correct.

(i) What is the probability of getting 90% with random guessing? **1**

(ii) What is the probability of getting at most 90% with random guessing? **2**

(d) A particle P is moving in simple harmonic motion. At a time t seconds, its acceleration is given by $\dfrac{d^2x}{dt^2} = -9(x-2)$, where x metres is the displacement from the origin O. Initially the particle is at O and its velocity is 8 m/s.

(i) Find the centre and period of the motion. **2**

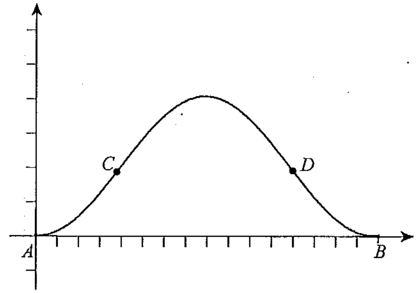
(ii) Show that $v^2 = 64 + 36x - 9x^2$, where v m/s is the velocity of P. **2**

(iii) Find the maximum speed of the particle. **2**

Marks

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The graph of the function $f(x) = 2\sin^2 x$ is shown below:



(i) Determine the coordinates of A and B. **1**

(ii) Find $f''(x)$, and hence determine the coordinates of C and D, the points where the gradient to the curve is a maximum. **2**

(iii) Calculate $\displaystyle\int_A^B (2\sin^2 x)\,dx$, leaving your answer in terms of π. **2**

(iv) Show that $\sin^4 x = \dfrac{3}{8} - \dfrac{1}{2}\cos 2x + \dfrac{1}{8}\cos 4x$. **2**

(v) Hence, determine the volume of the solid of revolution, when $y = 2\sin^2 x$ is rotated about the x-axis between A and B. Leave your answer in terms of π. **2**

(b) (i) Show that $\cos x - \sin x = \sqrt{2}\cos\left(x + \dfrac{\pi}{4}\right)$. **1**

(ii) Prove that $\dfrac{d}{dx}(e^x \cos x) = \sqrt{2}e^x \cos\left(x + \dfrac{\pi}{4}\right)$. **1**

(iii) Prove by mathematical induction that if $y = e^x \cos x$, then **4**

$$\frac{d^n y}{dx^n} = (\sqrt{2})^n e^x \cos\left(x + \frac{n\pi}{4}\right).$$

End of paper

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Mathematics Extension 1

Solutions and marking guidelines

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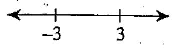
Section I

Sample answer

Question 1 D

$\frac{x^2}{x^2-9} > 0$ $\quad x^2 - 9 \neq 0$, $x \neq \pm 3$



If $-3 < x < 3$ then $\frac{x^2}{x^2-9} < 0$

$\therefore x < -3, x > 3$

Question 2 C

$\text{cosec}\left(-\frac{7\pi}{4}\right) = \frac{1}{\sin\left(\frac{\pi}{4}\right)}$

$= \sqrt{2}$

Question 3 A

$y = 2\sin^{-1}\left(\frac{x}{3}\right)$

$\frac{y}{2} = \sin^{-1}\left(\frac{x}{3}\right)$

$\therefore -\frac{\pi}{2} \le \frac{y}{2} \le \frac{\pi}{2}$ and $-1 \le \frac{x}{3} \le 1$

$-\pi \le y \le \pi$ and $-3 \le x \le 3$

$|y| \le \pi, |x| \le 3$

Question 4 C

Let $y = 3 - \frac{1}{2x+6}$

Interchange x and y:

$x = 3 - \frac{1}{2y+6}$

$\frac{1}{2y+6} = 3 - x$

$2y + 6 = \frac{1}{3-x}$

$\therefore 2y = -6 + \frac{1}{3-x}$

$y = -3 + \frac{1}{6-2x}$

$y = -3 - \frac{1}{2x-6}$

Question 5 B

$\int_0^{\frac{\pi}{2}} (\sin x)(\cos^4 x)\,dx$

Put $u = \cos x$

$du = -\sin x\,dx$

$\int_1^0 (u^4) \cdot -du = \int_0^1 u^4\,du$

Question 6 D

$y = \sin^{-1}\left(\frac{a}{x}\right)$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{a}{x}\right)^2}} \times \frac{-a}{x^2}$

$= \frac{-a}{x\sqrt{x^2-a^2}}$

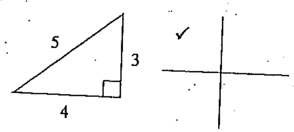
Question 7 A

$\int \frac{1}{c(a-bx)}\,dx = \frac{1}{bc}\int \frac{-b}{a-bx}\,dx$

$= -\frac{1}{bc}\ln(a-bx)$

Question 8 C

$\sin x = \frac{3}{5}$



$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

$= \frac{2\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2}$

$= -\frac{24}{7}$

Question 9 C

$f(x) = -x^2 + 2x + 3$

$= -(x^2 - 2x - 3)$

$= -(x+1)(x-3)$

$\therefore \frac{1}{f(x)} = \frac{-1}{(x+1)(x-3)}$

$\therefore$ asymptotes $x = -1, x = 3$

Question 10 B

$\frac{mx_2 + nx_1}{m+n} = 10$

$\frac{m(7) + n(-2)}{m+n} = 10$

$7m - 2n = 10m + 10n$

$-12n = 3m$

$\therefore \frac{m}{n} = -4$

$\therefore m : n = -4 : 1$ or $4 : -1$

Section II

Sample answer

Question 11

(a) (i) ${}^5C_3 \times {}^6C_1 = 60$

(ii) ${}^4C_2 \times {}^5C_0 = 6$

$\therefore P(E) = \frac{6}{60} = \frac{1}{10}$

(b) (i) $f(3) = 10 - e^2 > 0$

$f(4) = 17 - e^3 < 0$

$\therefore$ root lies between $x = 3$ and $x = 4$.

(ii) $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ $\quad f'(x) = 2x - e^{x-1}$

$= 3.5 - \frac{1.0675...}{-5.1825...}$

$= 3.71$

(c) $P(1) = 0 \quad \therefore (a+b) = 4 \quad (1)$

$P(-2) = 6 \quad \therefore -8a + 4b + 14 = 6$

$-2a + b = -2 \quad (2)$

$(1) - (2): 3a = 6$

$a = 2 \quad \therefore b = 2$

$\therefore P(x) = 2x^3 + 2x^2 - 6x + 2$

$= (x-1)(2x^2 + 4x - 2)$

$= (x-1)(x - (-1 - \sqrt{2}))(x - (-1 + \sqrt{2}))$

(d) (i) $\sin(\alpha + \beta) + \sin(\alpha - \beta)$

$= \sin\alpha\cos\beta + \sin\beta\cos\alpha + \sin\alpha\cos\beta - \sin\beta\cos\alpha$

$= 2\sin\alpha\cos\beta$

(ii) $\sin\gamma = 2\sin\alpha\cos\beta$

Using (i) above:

$= \sin(\alpha + \beta) + \sin(\alpha - \beta)$

$= \sin(\pi - \gamma) + \sin(\alpha - \beta)$

$\therefore \sin\gamma = \sin\gamma + \sin(\alpha - \beta)$

$(\alpha - \beta) = 0$

$\alpha - \beta = 0, \pi, 2\pi...$

$\alpha = \beta$, angles in a triangle are less than $180°$

$\therefore \Delta ABC$ is isosceles

Question 12

(a) $y = e^{2x} + e^{-2x}$

$y' = 2x^{2x} - 2e^{-2x}$

$y'' = 4e^{2x} + 4e^{-2x}$

$\therefore 4(e^{2x} + e^{-2x}) + 2a(e^{2x} - e^{-2x}) + b(e^{2x} + e^{-2x}) = 5e^{2x} + e^{-2}$

Equating coefficients:

$\therefore 2a + b = 1$ and $-2a + b = -3$

$\therefore 2b = -2$

$b = -1, a = 1$

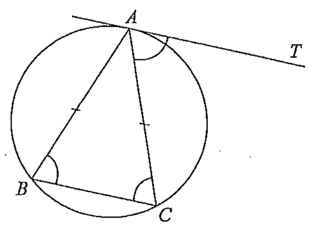
(b) $\int_0^{\pi} \frac{4}{\sqrt{16 - x^2}} dx = 4\left[\sin^{-1}\left(\frac{x}{4}\right)\right]_0^{\pi}$

$= 4\left[\sin^{-1}\left(\frac{\pi}{4}\right) - \sin^{-1}(0)\right]$

$= 4\left[\frac{1}{\sqrt{2}} - 0\right]$

$= \frac{4}{\sqrt{2}}$ or $2\sqrt{2}$

(c)



$\angle CAT = \angle ABC$ (angle between the tangent and the chord i equal to the angle in the alternate segment)

ΔABC is isosceles (two equal sides)

$\therefore \angle ACB = \angle ABC$ (base angles of an isosceles triangle are equal)

$\therefore \angle ACB = \angle CAT$ (both equal $\angle ABC$)

Alternate angles are equal only when lines are parallel.

$\therefore AD \parallel BC$

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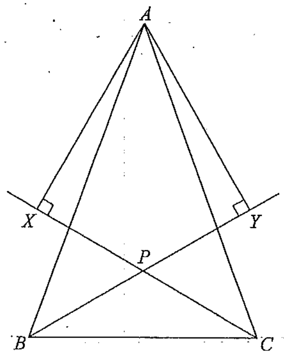
Sample answer

(d)



Since $AB = AC$

then $\angle ACB = \angle ABC$ (base angles of an isosceles Δ)

$\therefore \angle ACX = \angle ABY$ (half of equal angles)

In ΔAXC and ΔAYB:

$AC = AB$

$\angle ACX = \angle ABY$ (above)

$\angle AXC = \angle AYB$ (right angles)

$\therefore \Delta AXC \equiv \Delta AYB$ (AAS)

$\therefore AX = AY$ (matching sides in congruent triangles)

Question 13

(a) (i) $\frac{y}{x} = \frac{\frac{2at^2}{t^2+1}}{\frac{2at}{t^2+1}}$

$= \frac{2at^2}{t^2+1} \times \frac{t^2+1}{2at}$

$= t$, as required

(ii) $x = \frac{2a\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)^2 + 1}$

$x = \frac{\frac{2ay}{x}}{\frac{y^2}{x^2} + 1}$

Multiple numerator and denominator by x^2.

$x = \frac{2ayx}{y^2 + x^2}$

$x^2 + y^2 = 2ay$

$x^2 + y^2 - 2ay = 0$

Completing the squares,

$(x-0)^2 + (y-a)^2 = a^2$, i.e. a circle.

$C(0, a)$ and radius $= a$

(b) $\left(2x - \frac{1}{x}\right)^{11}$

Coefficient of x^5: ${}^{11}C_k\,(2x)^{11-k}\left(-\frac{1}{x}\right)^k$

$\therefore (x^{11-k})(x^{-k}) = x^5$

$11 - 2k = 5$

$2k = 6$

$k = 3$

$\therefore$ Coefficient of $x^5 = {}^{11}C_3\, 2^{11-3}(-1)^3$

$= -42\,240$

(c) (i) $P(90\%\text{ score}) = {}^{10}C_9\left(\frac{1}{4}\right)^9\left(\frac{3}{4}\right)^1$ or $\frac{15}{524\,288}$

(ii) $P(\text{at most }90\%) = 1 - P(100\%)$

$= 1 - \left[{}^{10}C_{10}\left(\frac{1}{4}\right)^{10}\left(\frac{3}{4}\right)^0\right]$

$= 1 - \frac{1}{4^{10}}$ or $\frac{4^{10} - 1}{4^{10}}$

$= 0.999999046$

(d) (i) $\ddot{x} = -9(x-2)$

$\therefore n^2 = 9$ and the centre of motion is 2

$n = 3$

i.e. $T = \frac{2\pi}{3}$ and $x = 2$

Sample answer

(ii) $\ddot{x} = -9(x-2)$

$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9(x-2)$

$\frac{1}{2}v^2 = \frac{-9(x-2)^2}{2} + C$

$t = 0,\ x = 0,\ v = 8$

$\therefore C = 50$

$\therefore \frac{1}{2}v^2 = \frac{-9(x-2)^2}{2} + 50$

$v^2 = 64 + 36x - 9x^2$

(iii) Maximum velocity when at centre of motion (i.e. $x = 2$)

$\therefore v^2 = 64 + 36(2) - 9(2)^2$

$v^2 = 100$

$v = \pm 10\ \text{ms}^{-1}$

i.e. v_{max} is 10 m/s

Question 14

(a) (i) $f(x) = 2\sin^2 x$

$\therefore$ At A and B $f(x) = 0$

$2\sin^2 x = 0$

$\sin^2 x = 0$

$x = 0, \pi, 2\pi, \ldots$

$\therefore A(0, 0)$ and $B(\pi, 0)$

(ii) $f'(x) = 4\sin x \cos x$

$\therefore f'(x) = 2\sin 2x$

and $f''(x) = 4\cos 2x$

Now, $f''(x) = 0$

$\therefore 4\cos 2x = 0$

$\cos 2x = 0$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$

$\therefore C\left(\frac{\pi}{4}, 1\right)$ and $D\left(\frac{3\pi}{4}, 1\right)$

(iii) $\int_0^{\pi} 2\sin^2 x\,dx = 2\int_0^{\pi}\frac{1}{2}(1 - \cos 2x)\,dx$

$= \left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$

$= (\pi - 0) - (0 - 0)$

$= \pi$

(iv) $\sin^4 x = \left[\frac{1}{2}(1 - \cos 2x)\right]^2$

$= \frac{1}{4}[1 - 2\cos 2x + \cos^2 2x]$

$= \frac{1}{4}\left[1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right]$

$= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$

(v) $V = \pi\int_0^{\pi}[2\sin^2 x]^2\,dx$

$= 4\pi\int_0^{\pi}\sin^4 x\,dx$

$= 4\pi\int_0^{\pi}\left[\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right]dx$

$= 4\pi\left[\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x\right]_0^{\pi}$

$= 4\pi\left[\left(\frac{3\pi}{8}\right) - (0)\right]$

$= \frac{3\pi^2}{2}$ units^3

(b) (i) $\cos x - \sin x = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)$

$= \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$

(ii) $\frac{d}{dx}(e^x\cos x) = e^x\cos x + e^x(-\sin x)$

$= e^x(\cos x - \sin x)$

$= e^x\sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$

Sample answer

(iii) $\frac{d^n y}{dx^n} = (\sqrt{2})^n e^x\cos\left(x + \frac{n\pi}{4}\right)$

S_1 : $n = 1$ $\frac{dy}{dx} = \sqrt{2}e^x\cos\left(x + \frac{\pi}{4}\right)$ as above in (ii)

S_2 : Assume true for $n = k$

i.e. $\frac{d^k y}{dx^k} = (\sqrt{2})^k e^x\cos\left(x + \frac{k\pi}{4}\right)$

S_3 : Show true for $n = k + 1$

i.e. $\frac{d^{k+1}y}{dx^{k+1}} = (\sqrt{2})^{k+1}e^x\cos\left(x + \frac{(k+1)\pi}{4}\right)$

Now,

$\frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$

$= \frac{d}{dx}\left[(\sqrt{2})^k e^x\cos\left(x + \frac{k\pi}{4}\right)\right]$

$= (\sqrt{2})^k\left[e^x\cos\left(x + \frac{k\pi}{4}\right) + e^x\left(-\sin\left(x + \frac{k\pi}{4}\right)\right)\right]$

$= (\sqrt{2})^k e^x\left[\cos\left(x + \frac{k\pi}{4}\right) - \sin\left(x + \frac{k\pi}{4}\right)\right]$

$= (\sqrt{2})^k e^x\sqrt{2}\left[\frac{1}{\sqrt{2}}\cos\left(x + \frac{k\pi}{4}\right) - \frac{1}{\sqrt{2}}\sin\left(x + \frac{k\pi}{4}\right)\right]$

$= (\sqrt{2})^{k+1}e^x\cos\left(\left(x + \frac{k\pi}{4}\right) + \frac{\pi}{4}\right)$

$= (\sqrt{2})^{k+1}e^x\cos\left(x + (k+1)\frac{\pi}{4}\right)$

$\therefore$ If true for $n = k$, then true for $n = k + 1$.

S_4 : Since true for $n = 1$, then true for $n = 1 + 1 = 2$ and so on for all integral values of n.

Syllabus outcomes and marking guide

HE2, Band 6

- Completes the proof correctly 4
- Simplifies to show $\frac{d^{k+1}y}{dx^{k+1}}$ 3
- Finds the derivative of $\frac{d^k y}{dx^k}$ correctly 2
- Shows true for $n = 1$ 1