PENRITH HIGH SCHOOL

MATHEMATICS EXTENSION 2
2012

HSC Trial

Assessor: Mr Ferguson

General Instructions:

- Reading time – 5 minutes
- Working time – **3 hours**
- Write using black or blue pen. Black pen is preferred
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- A multiple choice answer sheet is provided at the back of this paper.
- Show all necessary working in Questions 11 – 16.
- Work on this question paper will not be marked.

Total marks – 100

SECTION 1 – Pages 2 – 5
10 marks
- Attempt Questions 1 – 10
- Allow about 15minutes for this section.

SECTION 2 – Pages 6 – 12
90 marks
- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.

Section1

Question	Mark
1	
2	
3	
4	
5	

Question	Mark
6	
7	
8	
9	
10	
Total	/10

Section 2

Question	Mark
11	/15
12	/15
13	/15
14	/15
15	/15
16	/15

Total	/100
%	

This paper MUST NOT be removed from the examination room

Student Number: ....

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

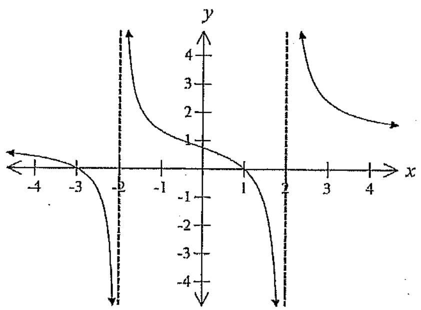
$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

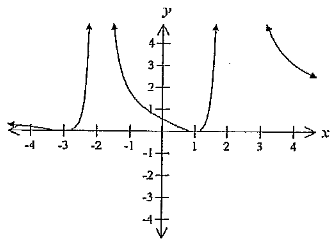
SECTION 1: Circle the correct answer on the multiple choice answer sheet

1 The diagram shows the graph of the function $y = f(x)$.

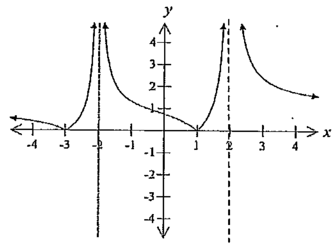


Which of the following is the graph of $y=|f(x)|$?

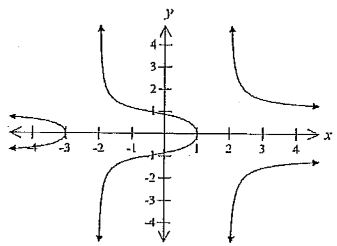
(A)



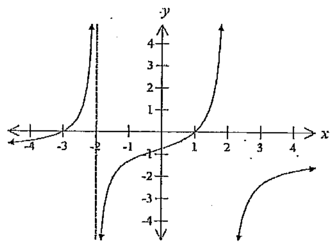
(B)



(C)



(D)



2 Let $z=4+i$. What is the value of $\overline{iz}$?

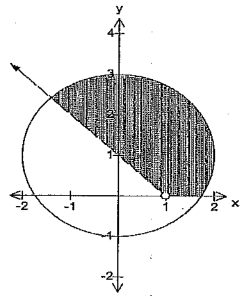
(A) $-1-4i$

(B) $-1+4i$

(C) $1-4i$

(D) $1+4i$

3 Consider the Argand diagram below.



Which inequality could define the shaded area?

(A) $|z-i|\leq 2$ and $0\leq \arg(z-1)\leq \frac{3\pi}{4}$

(B) $|z+i|\leq 2$ and $0\leq \arg(z-1)\leq \frac{3\pi}{4}$

(C) $|z-i|\leq 2$ and $0\leq \arg(z-1)\leq \frac{\pi}{4}$

(D) $|z+i|\leq 2$ and $0\leq \arg(z-1)\leq \frac{\pi}{4}$

4 Consider the hyperbola with the equation $\frac{x^2}{9}-\frac{y^2}{5}=1$.

What are the coordinates of the vertex of the hyperbola?

(A) $(\pm 3,0)$

(B) $(0,\pm 3)$

(C) $(0,\pm 9)$

(D) $(\pm 9,0)$

5 The points $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ lie on the same branch of the hyperbola $xy=c^2$ ($p\neq q$). The tangents at P and Q meet at the point T. What is the equation of the normal to the hyperbola at P?

(A) $p^2x-py+c-cp^4=0$

(B) $p^3x-py+c-cp^4=0$

(C) $x+p^2y-2c=0$

(D) $x+p^2y-2cp=0$

6 What is the value of $\int \sec x dx$? Use the substitution $t = \tan\frac{x}{2}$.

(A) $\ln|(t+1)(t-1)|+c$

(B) $\ln\left|\frac{1+t}{1-t}\right|+c$

(C) $\ln|(1+t)(1-t)|+c$

(D) $\ln\left|\frac{t+1}{t-1}\right|+c$

7 Let $I_n = \int_0^x \sin^n t\,dt$, where $0 \le x \le \frac{\pi}{2}$.

Which of the following is the correct expression for I_n ?

(A) $I_n = \left(\frac{n-1}{n}\right)I_{n-2}$ with $n \ge 2$.

(B) $I_n = \left(\frac{n+1}{n}\right)I_{n-2}$ with $n \ge 2$.

(C) $I_n = n(n-1)I_{n-2}$ with $n \ge 2$.

(D) $I_n = n(n+1)I_{n-2}$ with $n \ge 2$.

8 The region enclosed by $y = x^3$, $y = 0$ and $x = 2$ is rotated around the y-axis to produce a solid. What is the volume of this solid?

(A) $\frac{8\pi}{5}$ units^3

(B) $\frac{32\pi}{5}$ units^3

(C) $\frac{64\pi}{5}$ units^3

(D) $\frac{16\pi}{5}$ units^3

9 What is the angle at which a road must be banked so that a car may round a curve with a radius of 100 metres at 90 km/h without sliding? Assume that the road is smooth and gravity to be 9.8 ms^{-2}.

(A) 83°10′

(B) 32°32′

(C) 83°6′

(D) 32°53′

10 The polynomial equation $x^3+4x^2-2x-5=0$ has roots α, β and γ. Which of the following polynomial equations have roots α^2, β^2 and γ^2 ?

(A) $x^3-20x^2-44x-25=0$

(B) $x^3-20x^2+44x-25=0$

(C) $x^3-4x^2+5x-1=0$

(D) $x^3+4x^2+5x-1=0$

SECTION 2

Question 11 (15 marks) (Use a new page to write your answers)

(a) Find (i) $\int \frac{t^2-1}{t^3}\,dt$. 4

(ii) $\int \frac{dx}{\sqrt{6-x-x^2}}$

(b) Evaluate (i) $\int_0^1 \frac{x}{(x+1)(2x+1)}\,dx$ 3

(ii) $\int_0^{\frac{\pi}{4}} x\tan^2 x\,dx$ 3

(c) (i) If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$, show that for $n > 1$, 3

$I_n = (\frac{\pi}{2})^n - n(n-1)I_{n-2}$

(ii) Hence find the area of the finite region bounded by the curve 2

$y = x^4 \cos x$ and the x axis for $0 \le x \le \frac{\pi}{2}$.

Question 12 (15 marks) (Use a new page to write your answers)

(a) Given that $z = \sqrt{2} - \sqrt{2}i$ and $w = -\sqrt{2}$, find, in the form $x+iy$:

(i) wz^2 1

(ii) $\arg z$ 1

(iii) $\frac{z}{z+w}$ 2

(iv) $|z|$ 1

(v) z^{10} 2

(b) Find the values of real numbers a and b such that $(a+ib)^2 = 5-12i$ 2

(c) Draw Argand diagrams to represent the following regions 2

(i) $1 \le |z+4-3i| \le 3$

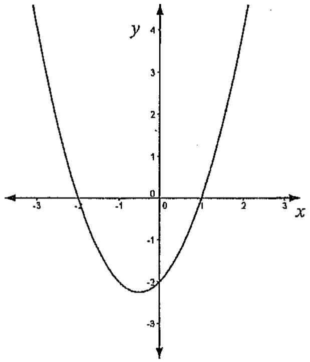
(ii) $\frac{\pi}{6} \le \arg z \le \frac{\pi}{3}$

(d) (i) Show that $\frac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta} = i\cot\frac{\theta}{2}$ 2

(ii) Hence solve $\left(\frac{z-1}{z+1}\right)^8 = -1$ 2

Question 13 (15 marks) (Use a new page to write your answers)

(a) The diagram shows the graph of the function $f(x)=x^2+x-2$. On separate diagrams sketch the following graphs, showing clearly any intercepts on the coordinate axes and the equations of any asymptotes.



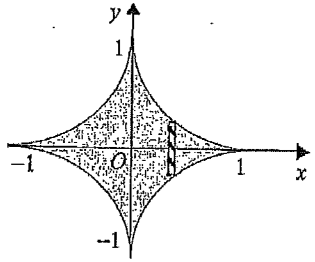
(i) $y=|f(x)|$ 1

(ii) $y=[f(x)]^2$ 1

(iii) $y=\dfrac{1}{f(x)}$ 2

(iv) $y=\log_e f(x)$ 2

(b) The horizontal base of a solid is the area enclosed by the curve $|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}=1$.
Vertcial cross sections taken perpendicular to the x-axis are equilateral triangles with one side in the base.



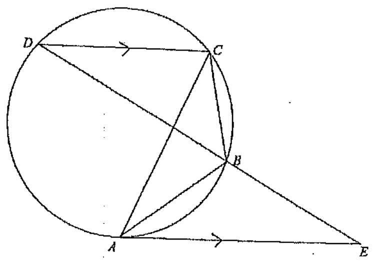
(i) Show that the volume of the solid is given by $V=2\sqrt{3}\int_0^1(1-\sqrt{x})^4\,dx$ 2

(ii) Use the substitution of $u=1-\sqrt{x}$ to evaluate this integral. 3

(c) The tangent AE is parallel to the chord DC.

(i) Prove that $(AB)^2=BC.BE$ 3

(ii) Hence or otherwise prove that $\dfrac{AC}{AE}=\sqrt{\dfrac{BC}{BE}}$ 1



Question 14 (15 marks) (Use a new page to write your answers)

(a) The equation of an ellipse is given by $4x^2+9y^2=36$.

(i) Find S and S' the foci of the ellipse 2

(ii) Find the equations of the directrices M and M' 1

(iii) Sketch the ellipse showing foci, directrices and axial intercepts. 2

(iv) Let P be any point on the ellipse. 2
Show $SP+S'P=6$

(v) Find the equation of the chord of contact from an external point $(3,2)$ 1

(b) (i) Sketch the rectangular hyperbola $xy=c^2$, labelling the point $P\left(ct,\frac{c}{t}\right)$ on it. 1

(ii) Show that the equations of the tangent and normal to the hyperbola at P are $x+t^2y=2ct$ and $ty+ct^4=t^3x+c$ respectively. 3

(iii) If the tangent at P meets the coordinate axes at X and Y respectively and the normal at P meets the lines $y=x$ and $y=-x$ at R and S respectively, prove that the quadrilateral $RYSX$ is a rhombus. 3

Question 15 (15 marks) (Use a new page to write your answers)

(a) When a certain polynomial is divided by $x+1$, $x-3$ the respective remainders are 6 and -2. Find the remainder when this polynomial is divided by x^2-2x-3. 3

(b) The cubic equation $x^3+px+q=0$ has 3 non-zero roots $\alpha,\ \beta,\ \gamma$. 3

Find, in terms of the constants p,q the values of

(i) $\alpha^2+\beta^2+\gamma^2$

(ii) $\alpha^3+\beta^3+\gamma^3$.

(c) If $\alpha,\ \beta,\ \gamma$ are the roots of the equation $3x^3-5x^2-4x+3=0$, find the cubic equation with roots $\alpha-1, \beta-1, \gamma-1$. 3

(d) A polynomial of degree n is given by $P(x)=x^n+ax-b$. It is given that the polynomial has a double root at $x=\alpha$.

(i) Find the derived polynomial $P'(x)$ and show that $\alpha^{n-1}=-\frac{a}{n}$. 3

(ii) Show that $\left(\frac{a}{n}\right)^n+\left(\frac{b}{n-1}\right)^{n-1}=0$. 2

(iii) Hence deduce that the double root is $\frac{bn}{a(n-1)}$. 1

Question 16 (15 marks) (Use a new page to write your answers)

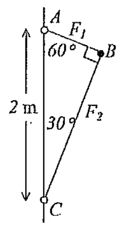
(a) For $a>0$, $b>0$, $c>0$ and $d>0$ and given that $\dfrac{a+b}{2}\geq\sqrt{ab}$, show that $\dfrac{a+b+c+d}{4}\geq\sqrt[4]{abcd}$ 2

(b) (i) Use De Moivre's theorem to express $\tan 5\theta$ in terms of powers of $\tan\theta$. 3

(ii) Hence show that $x^4-10x^2+5=0$ has roots $\pm\tan\dfrac{\pi}{5}$ and $\pm\tan\dfrac{2\pi}{5}$. 2

(iii) Deduce that $\tan\dfrac{\pi}{5}.\tan\dfrac{2\pi}{5}.\tan\dfrac{3\pi}{5}.\tan\dfrac{4\pi}{5}=5$ 1

(c) A mass 10 kg, centre B is connected by light rods to sleeves A and C which revolve freely about the vertical axis AC but do not move vertically.



(i) Given $AC=2$ metres, show that the radius of the circular path of rotation of B is $\dfrac{\sqrt{3}}{2}$ metres. 1

(ii) Find the tensions in the rods AB, BC when the mass makes 90 revolutions per minute about the vertical axis. 3

(d) Given that $a_n=\sqrt{2+a_{n-1}}$ for integers $n\geq1$ and $a_0=1$, by mathematical induction prove that for $n\geq1$: 3

$$\sqrt{2}<a_n<2$$

Section 1

1) B

2) $iz = i(4+i)$

$= 4i + i^2$

$= 4i - 1$

$= -1 + 4i$

$\overline{iz} = -1 - 4i$

$= A$

3) A

4) Let $y = 0$

$\frac{x^2}{9} = 1$

$x^2 = 9$

$x = \pm 3$

$(\pm 3, 0)$

A

5) Normal and $P(cp, \frac{c}{p})$

$\therefore y - \frac{c}{p} = p^2(x - cp)$

$py - c = p^3(x - cp)$

$py - c = p^3 x - p^4 c$

$p^3 x - py + c - cp^4 = 0$

$x = ct \quad \frac{dx}{dt} = c$

$y = \frac{c}{t} \quad \frac{dy}{dt} = -\frac{c}{t^2}$

$\therefore \frac{dy}{dx} = -\frac{1}{t^2}$

$\therefore$ gradient of normal is t^2

in this case is p^2

B

6) $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}(1+t^2)$

$dx = \frac{2\,dt}{1+t^2}$

$\int \sec x = \int \frac{1+t^2}{1-t^2} \cdot \frac{2}{1+t^2}\,dt$

$= \int \frac{2}{1-t^2}\,dt$

$= \int \frac{A}{1-t} + \frac{B}{1+t}\,dt$

$A(1+t) + B(1-t) = 2$

Let $t = -1 \quad \therefore B = 1$

$t = 1 \quad \therefore A = 1$

$\therefore \int \frac{1}{1-t} + \int \frac{1}{1+t} = -\ln(1-t) + \ln(1+t) = \ln\left|\frac{1+t}{1-t}\right| + C$. B

7) $I_n = \int_0^x \sin^n x\,dx \quad 0 \le x \le \frac{\pi}{2}$

$\int_0^x \overset{u}{\sin^{n-1} x}\ \overset{v'}{\sin x}$

$u = \sin^{n-1} x \quad u' = (n-1)\sin^{n-2} x \cos x$

$v' = \sin x \quad v = -\cos x$

$I_n = \left[-\cos x \sin^{n-1} x\right] + n-1 \int \cos^2 x \sin^{n-2} x$

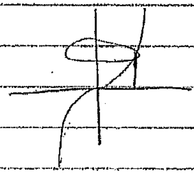
$I_n = (n-1)\int (1 - \sin^2 x)\sin^{n-2} x\,dx$

$= (n-1)\int \sin^{n-2} x - (n-1)\int \sin^n x\,dx$

$I_n(1 + n - 1) = (n-1)\int \sin^{n-2} x\,dx$

$n I_n = n - 1\, I_{n-2}$

$I_n = \frac{n-1}{n}(I_{n-2})$ A

8) 

$\int_0^2 \pi x\, y\,dx$

$= \int_0^2 \pi x\, x^3\,dx$

$= \int_0^2 \pi x^4\,dx$

$\pi\left[\frac{x^5}{5}\right]_0^2$

$= \frac{32\pi}{5}$ B

9) $\tan\theta = \frac{v^2}{rg}$

$= \frac{25^2}{100 \times 9.8}$

$= 0.6377$

$\theta = 32°32'$ B

10)

~~A~~ $\alpha\beta\gamma$ satisfy $x^3+4x^2-2x-5=0$

$\alpha^2\beta^2\gamma^2$ satisfy $(x^{\frac{1}{2}})^3+4(x^{\frac{1}{2}})^2-2(x^{\frac{1}{2}})-5=0$

$x^{\frac{3}{2}}+4x-2x^{\frac{1}{2}}-5=0$

$x^{\frac{3}{2}}-2x^{\frac{1}{2}}=-4x+5$

$x^{\frac{1}{2}}(x-2)=-4x+5$

$x(x-2)^2=(-4x+5)^2$

$x(x^2-4x+4)=16x^2-40x+25$

$x^3-4x^2+4x=16x^2-40x+25$

$x^3-20x^2+44x-25=0$ R

Section 2. Question 11

a)(i) $\int \frac{t^2-1}{t^3}\,dt$

$\int \frac{t^2}{t^3}-\int\frac{1}{t^3}$

$\int\frac{1}{t}-\int t^{-3}$

$\ln t-\frac{t^{-2}}{-2}$

$\ln t+\frac{1}{2t^2}+c$

(ii) $\int\frac{dx}{\sqrt{6-x-x^2}}$

$\int\frac{dx}{\sqrt{-(x^2+x-6)}}$

$\int\frac{dx}{\sqrt{-[(x+\frac{1}{2})^2-\frac{25}{4}]}}$

$=\int\frac{dx}{\sqrt{\frac{25}{4}-(x+\frac{1}{2})^2}}$

$=\sin^{-1}\frac{x+\frac{1}{2}}{\frac{5}{2}}$

$=\sin^{-1}\left(\frac{2x+1}{5}\right)+c.$

b) i $\int_0^1\frac{x}{(x+1)(2x+1)}$ $\qquad \frac{A}{x+1}+\frac{B}{2x+1}=x$

$A(2x+1)+B(x+1)=x$

let $x=-1$ $\quad -A=-1$

$A=1$

$x=-\frac{1}{2}$ $\quad \frac{1}{2}B=-\frac{1}{2}$

$B=-1$

$\int_0^1\frac{1}{x+1}+\int_0^1\frac{-1}{2x+1}$

$=\int_0^1\frac{1}{x+1}-\frac{1}{2}\int_0^1\frac{2}{2x+1}$ $\quad \left[\ln(x+1)-\frac{1}{2}\ln(2x+1)\right]_0^1$

$(\ln 2-\frac{1}{2}\ln 3)-(\ln 1-\cancel{\frac{1}{2}\ln 1})$

$=\ln\left(\frac{2}{\sqrt{3}}\right)$ $\quad \ln\frac{2\sqrt{3}}{3}$

(ii) $\int_0^{\frac{\pi}{4}} x\tan^2 x\,dx$ $\quad u=x \quad \frac{du}{dx}=1$

$\int_0^{\frac{\pi}{4}} x(\sec^2 x-1)\,dx$ $\quad \frac{dv}{dx}=\sec^2 x-1 \quad V=\tan x - x$

$\therefore I = \left[x(\tan x - x)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}(\tan x - x)\,dx$

$= \left[x(\tan x - x)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} - x\,dx$

$= \left[x(\tan x - x) + \ln(\cos x) + \frac{x^2}{2}\right]_0^{\frac{\pi}{4}}$

$= \left[x\tan x - \frac{x^2}{2} + \ln(\cos x)\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4} - \frac{\pi^2}{32} + \ln\frac{1}{\sqrt{2}}$

c $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$

$u = x^n \quad \frac{du}{dx} = nx^{n-1}$

$\frac{dv}{dx} = \cos x \quad V = \sin x$

$\left[x^n \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, n x^{n-1}\,dx.$

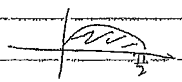
$\left(\frac{\pi}{2}\right)^n + n\int_0^{\frac{\pi}{2}} x^{n-1}\sin x\,dx$ $\quad u = x^{n-1} \quad \frac{du}{dx} = n-1x^{n-2}$

$\frac{dv}{dx} = \sin x \quad V = -\cos x$

$\left(\frac{\pi}{2}\right)^n + n\underbrace{\left[x^{n-1}\cdot\cos x\right]_0^{\frac{\pi}{2}}}_{=0} - n\int_0^{\frac{\pi}{2}} \cos x (n-1)x^{n-2}\,dx.$

$= \left(\frac{\pi}{2}\right)^n + 0 - n(n-1)\, I_{n-2}$

$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) I_{n-2}$

(ii) 

$A = I_4$

$= \left(\frac{\pi}{2}\right)^4 - 4\times 3\left[\left(\frac{\pi}{2}\right)^2 - 2\int_0^{\frac{\pi}{2}} \cos x\,dx\right]$

$= \left(\frac{\pi}{2}\right)^4 - 3\pi^2 + 24\left[\sin x\right]_0^{\frac{\pi}{2}}$

$= \left(\frac{\pi}{2}\right)^4 - 3\pi^2 + 24$

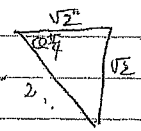
Question 12

a(i) $-\sqrt{2}(\sqrt{2}-\sqrt{2}i)^2$

$= -\sqrt{2}(2 - 4i + 2i^2)$

$= -\sqrt{2}(0 - 4i)$

$= 4\sqrt{2}i$

(ii) 

$= -\frac{\pi}{4}$

(iii) $\frac{z}{z+w}$

$\frac{\sqrt{2}-\sqrt{2}i}{\sqrt{2}-\sqrt{2}i+-\sqrt{2}}$

$= \frac{\sqrt{2}-\sqrt{2}i}{-\sqrt{2}i} \times \frac{\sqrt{2}i}{\sqrt{2}i}$

$= \frac{2i-2i^2}{-2i^2}$

$= \frac{2i+2}{2}$

$= 1+i$

(iv) $|z|$

$= \sqrt{\sqrt{2}^2+\sqrt{2}^2}$

$= \sqrt{4}$

$= 2$

(v) $z^{10} = 2\left(\text{cis} -\frac{\pi}{4}\right)^{10}$

$= 2^{10}\text{cis} -\frac{10\pi}{4} = 1024\,\text{cis} -\frac{2\pi}{4} = \text{cis} -\frac{\pi}{2} = -i1024$

b) $(a+ib)^2 = 5-12i$

$a^2+2abi-b^2 = 5-12i$

$a^2-b^2 = 5 \quad -(1)$

$2ab = -12 \quad -(2)$

$ab = -6$

$b = -\frac{6}{a}$

$a^2 - \frac{36}{a^2} = 5$

$a^4 - 5a^2 - 36 = 0$

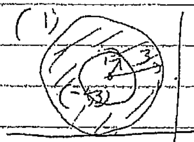
$(a^2-9)(a^2+4) = 0$

$a^2 = 9 \quad \therefore \boxed{a = \pm 3}$

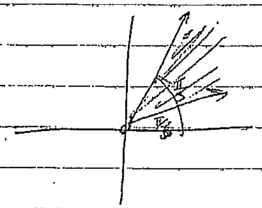
$2(\pm 3)b = -12$

$\therefore \boxed{b = (\mp 2)}$ $\quad a=3 \; b=-2$ or $a=-3 \; b=+2$

c) (i)



(ii)



d(i) $\frac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta} = \frac{2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2} - i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$= \frac{2\cos\frac{\theta}{2}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})}{2\sin\frac{\theta}{2}(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2})}$

$= \frac{\cos\frac{\theta}{2}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})}{-i\sin\frac{\theta}{2}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})}$

since $\sin x - i\cos x = -i(\cos x + i\sin x)$

$= i\cot\frac{\theta}{2}$ since $\frac{1}{-i} = i$

$\left(\frac{z-1}{z+1}\right)^8 = -1 \Rightarrow \frac{z-1}{z+1} = \sqrt[8]{-1}$

$\frac{z-1}{z+1} = \sqrt[8]{\operatorname{cis}(\pi+2k\pi)} = \operatorname{cis}\frac{(2k+1)\pi}{8} \quad k = 0, \pm 1, \pm 2$

$z-1 = \left(\operatorname{cis}\frac{(2k+1)\pi}{8}\right)(z+1)$

$z-1 = \operatorname{cis}\left(\frac{2k+1}{8}\right)\pi \, z + \operatorname{cis}\frac{(2k+1)\pi}{8}$

$z\left(1 - \operatorname{cis}\frac{(2k+1)\pi}{8}\right) = \operatorname{cis}\frac{(2k+1)\pi}{8} + 1$

$\therefore z = \frac{1+\operatorname{cis}\frac{(2k+1)\pi}{8}}{1-\operatorname{cis}\frac{(2k+1)\pi}{8}}$

$= i\cot\frac{(2k+1)\pi}{16}$ from (i)

$= i\cot\frac{\pm\pi}{16}, \; i\cot\frac{\pm 3\pi}{16}$

$= \pm i\cot\frac{\pi}{16}, \; \pm i\cot\frac{3\pi}{16}$ since $\cot x$ is an odd function

d(i) Alternatively Let $t = \tan\frac{\theta}{2}$.

L.H.S $\frac{1+\frac{1-t^2}{1+t^2} + \frac{i2t}{1+t^2}}{1-\frac{1-t^2}{1+t^2} - \frac{i2t}{1+t^2}}$

$= \frac{2+i2t}{2t^2-i2t}$

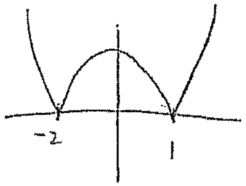
$= \frac{1+it}{t^2-it}$

$= \frac{i(t-i)}{t(t-i)}$

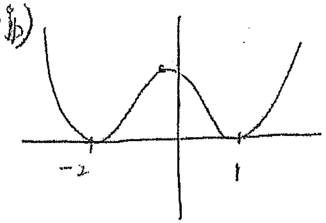
$= \frac{i}{t} = i\cot\frac{\theta}{2} = $ R.H.S

Question 13

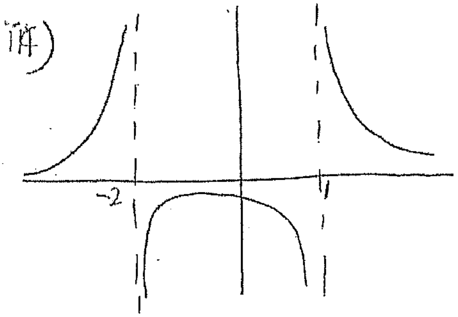
i) $y = |f(x)|$



ii)



iii)



(iv) $y = \log f(x)$

As $x \to -2$ or 1 $\quad f(x) \to 0 \quad \log f(x) \to +\infty$

crosses x axis when $\ln f(x) = 0$

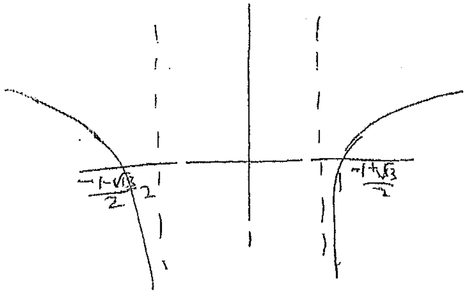
ie when

$f(x) = 1$

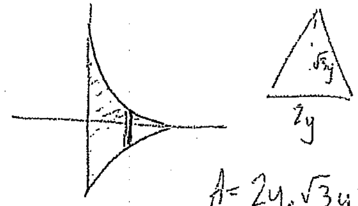
$x^2 + x - 2 = 1$

$x^2 + x - 3 = 0$

$x = \dfrac{-1 \pm \sqrt{13}}{2}$



(b) (i)



$A = 2y \cdot \sqrt{3}y \times \frac{1}{2}$

$A = \sqrt{3}y^2$

$\delta V = \sqrt{3}y^2 \delta x$

$V = \sum_{\delta x \to 0}^{\infty} \sqrt{3}y^2 \delta x$ $\qquad y^{\frac{1}{2}} = 1 - x^{\frac{1}{2}}$

$= \int_0^1 \sqrt{3}y^2\,dx$ $\qquad y = (1 - x^{\frac{1}{2}})^2$

$\qquad y^2 = (1 - x^{\frac{1}{2}})^4$

$= \int_0^1 \sqrt{3}y^2\,dx$

$= \sqrt{3}\int_0^1 (1 - \sqrt{x})^4\,dx$ $\qquad$ double since both sides of y axis

$= 2\sqrt{3}\int_0^1 (1 - \sqrt{x})^4\,dx$

(ii) $u = 1 - \sqrt{x}$ $\qquad x = (1-u)^2$ $\qquad x = 0 \quad u = 1$

$dx = -2(1-u)\,du$ $\qquad x = 1 \quad u = 0$

$= 2\sqrt{3}\int_1^0 u^4 \times -2(1-u)\,du$

$= 4\sqrt{3}\int_0^1 u^4 - u^5\,du$

$= 4\sqrt{3}\left[\frac{1}{5}u^5 - \frac{1}{6}u^6\right]_0^1$

$= 4\sqrt{3}\left[\frac{1}{5} - \frac{1}{6}\right]$

$= \frac{4\sqrt{3}}{30} = \frac{2\sqrt{3}}{15}$

C (i) Aim: prove $(AB)^2 = BC.BE$

proof: In ΔABC and ΔEBA

$\angle AEB = \angle CDE$ (alternate $\angle$'s on parallel lines)

$\angle CDE = \angle CAB$ (angles in the same segment)

$\therefore \angle AEB = \angle CAB$ — A

$\angle BAE = \angle BCA$ (angle in the alternate segment)

$\therefore \Delta ABC \,|||\, \Delta EBA$ equiangular

$\therefore \frac{AB}{BC} = \frac{BE}{AB}$ or $AB^2 = BC.BE$

(ii) $\frac{AC}{AE} = \frac{BC}{BA}$ since similar triangles have sides in proportion

$AB^2 = BC.BE$

$\therefore AB = \sqrt{BC.BE}$

$\frac{AC}{AE} = \frac{BC}{\sqrt{BC.BE}}$

$= \frac{BC}{\sqrt{BC}\sqrt{BE}}$

$\frac{AC}{AE} = \frac{\sqrt{BC}}{\sqrt{BE}}$

Question 14

(a) $4x^2 + 9y^2 = 36$

(i) $\frac{x^2}{9} + \frac{y^2}{4} = 1$

$a^2 = 9 \quad b^2 = 4 \quad b^2 = a^2(1-e^2)$

$4 = 9(1-e^2)$

$\frac{4}{9} = 1 - e^2$

$e^2 = \frac{5}{9}$

$e = \frac{\sqrt{5}}{3}$

$S(ae, 0) \quad S'(-ae, 0)$

$S(\sqrt{5}, 0) \quad S'(-\sqrt{5}, 0)$

(ii) $x = \pm \frac{a}{e}$

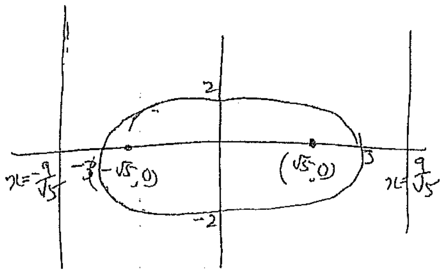
$= \pm \frac{3}{\frac{\sqrt{5}}{3}}$

$= \pm \frac{9}{\sqrt{5}}$

$M : x = \frac{9}{\sqrt{5}}$

$M' : x = -\frac{9}{\sqrt{5}}$

(iii)



(iv) $SP + SP' = 6$

$PS = e\,PM$
$P'S' = e\,P'M'$ where M and M' are the feet of the perpendiculars from P to M and m'.

$PS + P'S' = e(PM + P'M')$
$= e \cdot MM'$
$= e\left(\frac{a}{e} + \frac{a}{e}\right)$
$= \frac{2ae}{e}$
$PS + P'S' = 2a$

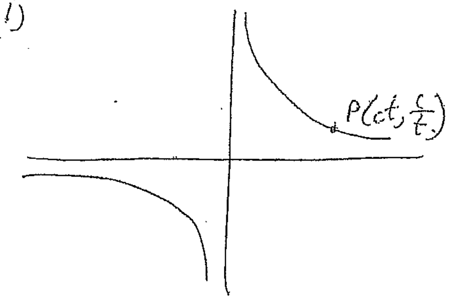
$a = 3$.
$\therefore SP + SP' = 6$

(v) $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$

$\frac{3x}{9} + \frac{2y}{4} = 1$

$\frac{x}{3} + \frac{y}{2} = 1.$

b(i)



(ii) $xy = c^2$
$y = \frac{c^2}{x}$
$\frac{dy}{dx} = -\frac{c^2}{x^2}$
at P, grad tangent $= -\frac{1}{t^2}$
$\therefore$ grad normal $= t^2$.

Eqn of tant $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$
$t^2y - ct = -x + ct$
$x + t^2y = 2ct$

Equation of normal
$y - \frac{c}{t} = t^2(x - ct)$
$ty - c = t^3x - ct^4$
$ty + ct^4 = t^3x + c$.

(iii) $\cancel{X}$ $(2ct, 0)$
$Y\left(0, \frac{2c}{t}\right)$
$R\left(\frac{c(t^2+1)}{t}; \frac{c(t^2+1)}{t}\right)$
$S\left(\frac{c(t^2-1)}{t}, \frac{-c(t^2-1)}{t}\right)$
Midpoint $XY = \left(ct, \frac{c}{t}\right)$
Midpoint $RS = \left(\frac{2ct^2}{t}, \frac{2c}{t}\right)$
$= ct, \frac{c}{t})$

Grad $XY = \frac{\frac{2c}{t}}{-2ct}$
$= -\frac{1}{t^2}$

Grad $RS = \frac{\frac{2ct^2}{t}}{\frac{2c}{t}}$
$= t^2$

$\therefore RS \perp XY$
$\therefore$ RYSX is a rhombus

Question 15

a) $P(x) = (x+1)(x-3)\cdot Q(x) + ax+b.$

$P(-1) = -a+b = 6$ — ①

$P(3) = 3a+b = -2$ — ②

① − ② $-4a = 8$

$a = -2$

$\therefore b = 4$

$x^2-2x-3 = (x+1)(x-3)$

when divided by x^2-2x-3

$\therefore R(x) = -2x+4$

b(i) α, β, γ $\therefore \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= 0 - 2p$

$= -2p.$

(ii) $\alpha^3+\beta^3+\gamma^3 =$

$\alpha^3 + p\alpha + q = 0$

$\beta^3 + p\beta + q = 0$

$\gamma^3 + p\gamma + q = 0$

$\alpha^3+\beta^3+\gamma^3 + p(\alpha+\beta+\gamma) + 3q = 0$

$\alpha^3+\beta^3+\gamma^3 + p.0 + 3q = 0$

$\therefore \alpha^3+\beta^3+\gamma^3 = -3q$

(c) Let $y = x-1$.

$\therefore x = y+1.$

$3(y+1)^3 - 5(y+1)^2 - 4(y+1) + 3 = 0$

$3(y^3+3y^2+3y+1) - 5(y^2+2y+1) - 4(y+1) + 3 = 0$

$3y^3+9y^2+9y+3-5y^2-10y-5-4y-4+3=0$

$3y^3+4y^2-5y-3=0$

in terms of x

$3x^3+4x^2-5x-3=0$

(d)(i) $P(x) = x^n + ax - b$

double root if $x = \alpha$

$P'(x) = nx^{n-1} + a$

$P''(x) = n(n-1)x^{n-2}$

note $P(\alpha)=0 \Rightarrow \alpha^n + a\alpha - b = 0$

$P'(\alpha) = 0 \Rightarrow n\alpha^{n-1} + a = 0$

$\therefore \alpha^{n-1} = -\frac{a}{n}$

(ii) $P(\alpha) = \alpha^n + a\alpha - b = 0$ — ①

$P'(\alpha) = n\alpha^{n-1} + a = 0$ — ②

$n\alpha^n + a\alpha = 0$ — α②

① − α② $(1-n)\alpha^n - b = 0$

$\alpha^n = \frac{b}{1-n}$ — ③

also $\alpha^{n-1} = -\frac{a}{n}$ — ④

from ③ $(\alpha^n)^{n-1} = \left(\frac{b}{1-n}\right)^{n-1}$

from ④ $(\alpha^{n-1})^n = \left(-\frac{a}{n}\right)^n$

$$\left(\frac{b}{1-n}\right)^{n-1} = \left(-\frac{a}{n}\right)^n$$

$$\left(-\frac{b}{n-1}\right)^{n-1} = (-1)^n\left(\frac{a}{n}\right)^n$$

$$(-1)^{n-1}\left(\frac{b}{n-1}\right)^{n-1} = (-1)^{n-1}\left(\frac{a}{n}\right)^n$$

$$\left(\frac{a}{n}\right)^n + \left(\frac{b}{n-1}\right)^{n-1} = 0$$

(iii) double root is α

$$\alpha = \frac{\alpha^n}{\alpha^{n-1}} \begin{matrix} \leftarrow \text{use } 3 \\ \leftarrow \text{use } 4 \end{matrix}$$

$$= \left(\frac{b}{1-n}\right) \Big/ -\frac{a}{n}$$

$$\alpha = \frac{bn}{-a(1-n)} = \frac{bn}{a(n-1)}$$

Question 16.

(a) Let $x = \frac{a+b}{2}$

$y = \frac{c+d}{2}$.

$\therefore \frac{x+y}{2} = \frac{a+b+c+d}{4}$

Now $\frac{x+y}{2} \geq \sqrt{xy}$

$\therefore \frac{a+b+c+d}{4} \geq \sqrt{\sqrt{ac}\sqrt{cd}}$

$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$

b) (i) $\text{cis}\,5\theta = (\text{cis}\,\theta)^5$

$\cos 5\theta + i\sin 5\theta = \cos^5\theta + i5\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - i10\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

$\therefore \cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$$\tan 5\theta = \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta \cdot \sin^2\theta + 5\cos\theta\sin^4\theta}$$

$$= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

$$= \tan\theta \cdot \frac{5 - 10\tan^2\theta + \tan^4\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

(ii) Let $x = \tan\theta$ then $\tan 5\theta = 0$ $\therefore x^4 - 10x^2 + 5 = 0$

$5\theta = 0$ or π or $2\pi, \ldots$

$\theta = \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$

$\tan\frac{3\pi}{5} = -\tan\left(\pi - \frac{3\pi}{5}\right)$

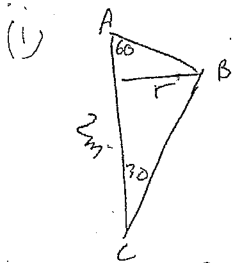
$= -\tan\frac{2\pi}{5}$ & $\tan\frac{4\pi}{5} = -\tan\frac{\pi}{5}$

root $x = \pm\tan\frac{\pi}{5}$ and $\pm\tan\frac{2\pi}{5}$, $\quad \alpha\beta\gamma\delta = \frac{e}{a}$

product of roots $\left(\tan\frac{\pi}{5}\right)\left(\tan\frac{2\pi}{5}\right)\tan\left(\frac{3\pi}{5}\right)\tan\left(\frac{4\pi}{5}\right) = 5 \quad = 5.$

Question 16(c)

(i)



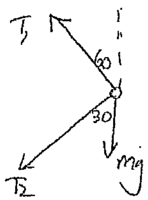
$\cos 60 = \frac{AB}{2}$

$AB = 1.$

$\sin 60 = \frac{r}{AB}$

$r = \frac{\sqrt{3}}{2}.$

(ii) Let tension in rods AB and BC be T_1 and T_2 respectively



$\sum F_V = 0$

$T_1 \cos 60 = T_2 \cos 30 + mg$

$T_1\left(\frac{1}{2}\right) = T_2\left(\frac{\sqrt{3}}{2}\right) + 10g.$

$T_1 = T_2\sqrt{3} + 20g$

$\sum F_H = mr\omega^2$

$T_1 \sin 60 + T_2 \sin 30 = mr\omega^2$

$T_1\left(\frac{\sqrt{3}}{2}\right) + T_2\left(\frac{1}{2}\right) = 10\left(\frac{\sqrt{3}}{2}\right)(3\pi)^2$

$T_1 = 90\pi^2 - \frac{T_2}{\sqrt{3}}$

$T_2\sqrt{3} + 20g = 90\pi^2 - \frac{T_2}{\sqrt{3}}$

$T_2 = \frac{5\sqrt{3}}{2}(9\pi^2 - 2g)$

$T_1 = \frac{5}{2}(27\pi^2 + 2g)$

so tensions in AB and BC are $\frac{5}{2}(27\pi^2+2g)$ N and $\frac{5\sqrt{3}}{2}(9\pi^2-2g)$ N

16d). $n = 1$

$a_1 < \sqrt{2 + a_0} = \sqrt{3}.$

since $\sqrt{2} < \sqrt{3} < 2$ is true for $n=1$.

Assume true for $n = k$

$\sqrt{2} < a_k < 2 \quad (A)$

Now prove true for $n = k+1$

$\sqrt{2} < a_{k+1} < 2 \quad (B)$

From (A) $\sqrt{2} < a_k < 2$

$2 + \sqrt{2} < 2 + a_k < 4$

$\sqrt{2+\sqrt{2}} < \sqrt{2 + a_k} < 2$

$\sqrt{2+\sqrt{2}} < a_{k+1} < 2$

Now $2 < 2 + \sqrt{2} \Rightarrow \sqrt{2} < \sqrt{2+\sqrt{2}}$

$\therefore \sqrt{2} < a_{k+1} < 2$

$\therefore$ proved by mathematical induction.