SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

AUGUST 2007
TRIAL HSC
YEAR 12

Mathematics Extension 2

General Instructions:

- Reading time—5 minutes.
- Working time—3 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may NOT be awarded for messy or badly arranged work.

Total marks—120 Marks

- Attempt questions 1–8.
- All questions are of equal value.
- Start each question in a separate answer booklet.

Examiner: Mr P. Bigelow

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

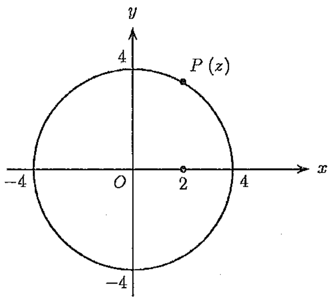
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1 (15 marks) — Marks

(a)



Copy the diagram onto your answer booklet.

Carefully indicate the position of the following:

(i) R representing $\bar{z}$, [1]

(ii) Q representing $-\frac{1}{2}z$, [1]

(iii) S representing $\dfrac{1}{z}$, [1]

(iv) T representing $\sqrt{z}$. [1]

(b) The complex number z is given by $z = -1 + i\sqrt{3}$.

(i) Show that $z^2 = 2\bar{z}$. [1]

(ii) Evaluate $|z|$ and $\arg z$. [1]

(iii) Show that z is a root of the equation $z^3 - 8 = 0$. [2]

(c) On an Argand diagram, shade the region where the inequalities [2]

$$0 \le \mathfrak{Re}(z) \le 4 \quad \text{and} \quad |z - 1 + i| \le 4 \quad \text{both hold.}$$

(d) If $z^2 = i$, find z in the form $a + ib$ where a and b are real. [2]

(e) Give reasons why each of the following statements is true or false. It is not necessary to evaluate the integrals.

(i) $\displaystyle\int_{-1}^{1} \frac{e^x - e^{-x}}{2}\,dx = 0.$ [1]

(ii) $\displaystyle\int_0^1 x^6\,dx < \int_0^1 x^7\,dx.$ [1]

(iii) $\displaystyle\int_0^{\pi} \sin^4 x\,dx > \int_0^{\pi} \sin 4x\,dx.$ [1]

Question 2 (15 marks)
(Start a new writing booklet) — Marks

(a) Evaluate $\displaystyle\int_0^1 \frac{dx}{(x+1)\sqrt{x+1}}$. [2]

(b) (i) Find a, b, and c such that [2]

$$\frac{x^2 + 5x - 4}{(x-1)(x^2+1)} \equiv \frac{a}{x-1} + \frac{bx + c}{x^2+1}.$$

(ii) Hence find $\displaystyle\int \frac{x^2 + 5x - 4}{(x-1)(x^2+1)}\,dx$. [2]

(c) Find $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$, using $x = \sec\theta$. [3]

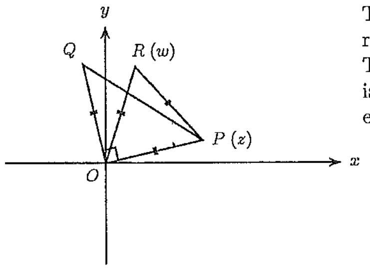
(d) (i) Show that $\displaystyle\sqrt{\frac{4-x}{4+x}} = \frac{4-x}{\sqrt{16-x^2}}$. [1]

(ii) Hence or otherwise find $\displaystyle\int_{-2}^{2} \sqrt{\frac{4-x}{4+x}}\,dx$. [2]

(e) Find $\displaystyle\int_0^1 2x\tan^{-1} x\,dx$. [3]

Question 3 (15 marks)
(Start a new writing booklet) — Marks

(a)



The point P in the Argand diagram represents the complex number z. The right-angled triangle OPQ is isosceles and the triangle OPR is equilateral.

(i) Find, in terms of z, the complex number represented by the point Q. [1]

(ii) Find, in terms of z, the complex number which represents the vector $\overrightarrow{QR}$. [2]

(iii) If R represents the complex number w, show that $w^3 + z^3 = 0$. [2]

(b) (i) Given that $y = x - \ln(\sec x + \tan x)$, $0 < x < \frac{\pi}{2}$. Show that $\frac{dy}{dx} = 1 - \sec x$. [1]

(ii) Hence show that $x < \ln(\sec x + \tan x)$ for $0 < x < \frac{\pi}{2}$. [3]

(c) It is known that $2 + i$ is a root of the equation $x^6 - 7x^4 + 31x^2 - 25 = 0$.

(i) Give a reason why $2 - i$ is also a root of the equation. [1]

(ii) Give a reason why $-(2 + i)$ is also a root of the equation. [2]

(iii) Find the other three roots, giving reasons (it should not be necessary to use long division). [3]

Question 4 (15 marks)
(Start a new writing booklet) — Marks

(a) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (where $a > b > 1$) has eccentricity $e = 1/2$.
The point (2, 3) lies on the ellipse.

(i) Find the values of a and b. [2]

(ii) Sketch the graph of the ellipse, showing clearly the intercepts on the axes, the coördinates of the foci, and the equations of the directrices. [2]

(b) (i) Show that $P\,(2\sqrt{2}\cos\theta,\ 3\sqrt{2}\sin\theta)$ lies on the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 2$. [1]

(ii) Show that the slope of the tangent at P is $-\frac{3\cos\theta}{2\sin\theta}$. [2]

(iii) Find the equation of the normal to the ellipse at P. [2]

(iv) Find the value of θ to the nearest degree if the normal passes through the point $(-2\sqrt{2}, 0)$. [2]

(c) At a dinner party there are twelve people, consisting of six married couples. Each of the women wears a different coloured scarf. The husband of each woman has a matching colour tie.

(i) The dinner takes place at a circular table. Find how many seating arrangements are possible if the women and men are in alternate positions. [2]

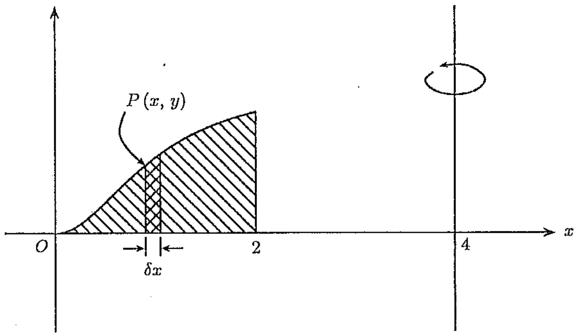
(ii) A committee of six is to be formed from the women and their partners, where not more than one of the six colours can be represented. How many such committees are possible? [2]

Question 5 (15 marks)
(Start a new writing booklet) Marks

(a) (i) Prove that for any polynomial $P(x)$, if k is a zero of multiplicity 2, then k is also a zero of $P'(x)$. [2]

(ii) Show that $x = 1$ is a double root of $x^{2n} - nx^{n+1} + nx^{n-1} - 1 = 0$. [2]

(b) The region shown in the diagram, bounded by the curve $y = \dfrac{x^2}{x^2+1}$, the x-axis, and the line $x = 2$, is rotated about about the line $x = 4$.



(i) Using the method of cylindrical shells, show that the volume δV of a shell distant x from the origin is given by: $\delta V \approx 2\pi(4-x)\left(1 - \dfrac{1}{1+x^2}\right)\delta x$. [2]

(ii) Hence find the volume of the solid. [3]

(c) An object of mass m kg is thrown vertically upwards. Air resistance is given by $R = 0.05\,mv^2$ where R is in Newtons and $v\,\text{ms}^{-1}$ is the speed of the object. (Take $g = 9.8\,\text{ms}^{-2}$.)

(i) Explain why the equation of motion is $\ddot{x} = -\left(\dfrac{196+v^2}{20}\right)$ where x is the height of the object t seconds after it is thrown. [2]

(ii) If the velocity of projection is $50\,\text{ms}^{-1}$, find the time taken to reach the highest point. [4]

Question 6 (15 marks)
(Start a new writing booklet) Marks

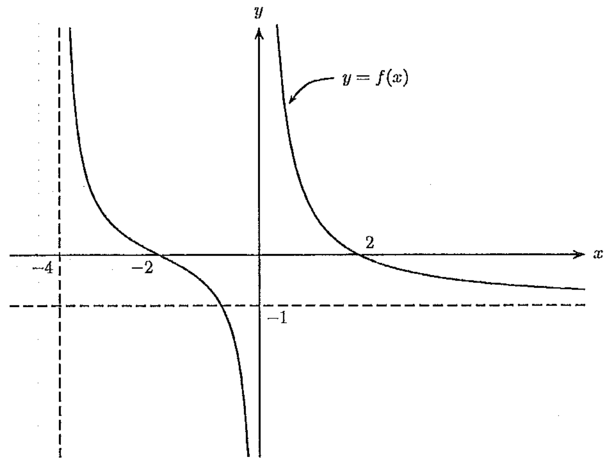
(a) Consider the curve $x^2 - xy + y^2 = 3$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{2x-y}{x-2y}$. [2]

(ii) Hence find the two stationary points on the curve. [2]

(iii) Find any values of x where there are vertical tangents. [1]

(b)



The sketch shows the graph of $y = f(x)$. There is a horizontal asymptote at $y = -1$ and vertical asymptotes at $x = 0$ and $x = -4$. Draw separate sketches of the following:

(i) $y = |f(x)|$ [1]

(ii) $y = \dfrac{1}{f(x)}$ [2]

(iii) $|y| = f(x)$ [2]

(iv) $y = [f(x)]^2$ [1]

(c) (i) By considering the perfect square $\left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^2$, show that $x + \dfrac{1}{x} \geq 2$. [2]

(ii) For all $a > 0$, $b > 0$, and $c > 0$, find the smallest possible values of

(α) $(a+b)\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$ [1]

(β) $(a+b+c)\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right)$ [1]

Question 7 (15 marks)

(Start a new writing booklet) **Marks**

(a) Let w be a non-real root of $z^7 - 1 = 0$.

(i) Show that $1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$. [1]

(ii) Show that $(1+w)(1+w^2)(1+w^4) = 1$. [1]

(iii) Form a quadratic equation with roots $(w + w^2 + w^4)$ and $(w^6 + w^5 + w^3)$. [2]

(iv) Sketch on an Argand diagram all seven roots of $z^7 - 1 = 0$. [1]

(b) (i) Show that if n is any even positive integer, [2]

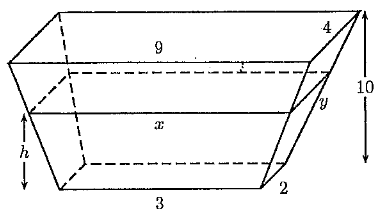
then $(1+x)^n + (1-x)^n = 2\sum_{k=0}^{n/2} \binom{n}{2k} x^{2k}$.

(ii) An alphabet consists of the three letters A, B, and C.

(α) Show that the number of words of five letters containing exactly two As is given by $\binom{5}{2} \times 2^3$. [1]

(β) Using (b)(i) and (ii)(α), or otherwise, show that if n is an even positive integer, then the number of words of n letters with zero or an even number of As is given by $\frac{1}{2}(3^n + 1)$. [3]

(c)



A solid has top and bottom faces which are parallel rectangles of dimensions 9×4 units and 3×2 units respectively. The altitude of the solid is 10 units.

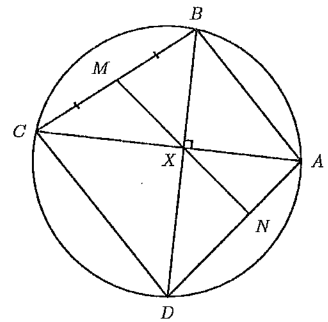
(i) A rectangle of dimensions x and y units is h units from the base. Assuming that x and y are linear functions of h, or otherwise, show that $x = \frac{3h}{5} + 3$ and $y = \frac{h}{5} + 2$. [2]

(ii) By considering a thin slice of, volume δV, thickness δh and dimensions $x \times y$ units, show that $\delta V = \left(\frac{3h}{5} + 3\right)\left(\frac{h}{5} + 2\right)\delta h$. Hence by integration find the volume V of the solid. [2]

Question 8 (15 marks)

(Start a new writing booklet) **Marks**

(a)



$ABCD$ is a cyclic quadrilateral. The diagonals AC and BD intersect at right-angles at X. M is the mid-point of BC. MX produced meets AD at N.

(i) Copy the diagram showing the above information.

(ii) Show that $M\widehat{B}X = M\widehat{X}B$. [2]

(iii) Show that MN is perpendicular to AD. [3]

(b) Consider the integral $I_n = \int_0^1 x^{2n+1} e^{-x^2}\, dx$.

It is given that $0 \le x^{2n+1} e^{-x^2} \le 1$ for $0 \le x \le 1$.

(i) Briefly explain why $0 \le I_n \le 1$. [1]

(ii) Use integration by parts to show that $I_n = -\frac{1}{2e} + nI_{n-1}$, for $n \ge 1$. [3]

(iii) Show that $I_0 = \frac{1}{2} - \frac{1}{2e}$. [2]

(iv) Prove by induction that, for all $n \ge 1$, [3]

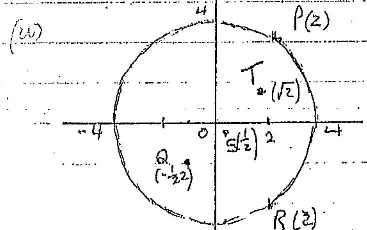
$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} = e - \frac{2eI_n}{n!}.$$

(v) Deduce that $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = e$. [1]

End of Paper

QUESTION 1

(a)



(b)(i) $z^2 = 1 - 2\sqrt{3}i - 3$

$= -2 - 2\sqrt{3}i$

$2\bar{z} = -2 - 2\sqrt{3}i = z^2$

(ii) $|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\text{Arg } z = \tan^{-1} - \sqrt{3}$

$= \frac{2\pi}{3}$

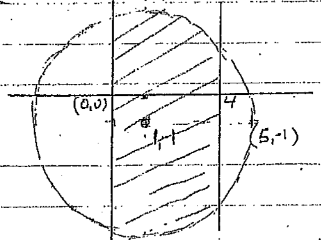
(iii) $z^3 = 2^3\left(\cos \frac{2\pi}{3} \cdot 3 + i \sin \frac{2\pi}{3} \cdot 3\right)$

$= 8$

$z^3 - 8 = 0$

z is a root of the eqn

(c) $\sqrt{(x-1)^2 + (y+1)^2} \le 4$



Circle Centre $(1,-1)$ $r = 4$.

(d) $i = (a+ib)^2 = a^2 + 2abi - b^2$

$a^2 - b^2 = 0 \quad ab = 1$ equating cfs

$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2$

$= 1$

① $a^2 + b^2 = 1$

② $a^2 - b^2 = 0$

①+② $2a^2 = 1$

$a = \pm \frac{1}{\sqrt{2}}$

① − ② $2b^2 = 1$

$b = \pm \frac{1}{\sqrt{2}}$

$z = \pm\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)$

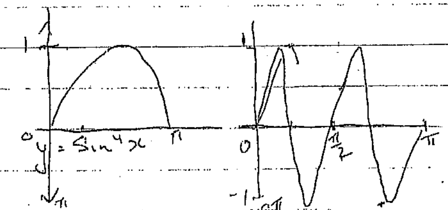
(e)(i) True

f(x) is odd with limits a, −a

(ii) False

for $0 < x \le \ \frac{1}{x^6} > \frac{1}{x^7}$

(iii) True



$\int_0^{\pi} \sin^4 x > 0 \qquad \int_0^{\pi} \sin 4x \, dx = 0$

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Question 2

(a) $I = \int_0^1 \frac{dx}{(x+1)\sqrt{x+1}}$

Let $u = x+1$; when $x = 1, u = 2$ and $X = 0, u = 1$

$du = dx$

$I = \int_1^2 u^{-3/2} du$

$= \left[\frac{u^{-1/2}}{-\frac{1}{2}}\right]_1^2$

$= \left[-2\left(\frac{1}{\sqrt{2}} - 1\right)\right]$

$= 2 - \sqrt{2}$

(b) (i) $\frac{x^2+5x-4}{(x-1)(x^2+1)} \equiv \frac{a}{x-1} + \frac{bx+c}{x^2+1}$

Multiplying both sides by the LHS denominator:

$x^2 + 5x - 4 \equiv a(x^2+1) + (bx+c)(x-1)$

$\equiv ax^2 + a + bx^2 - bx + cx - c$

$\equiv (a+b)x^2 + (c-b)x + (a-c)$

Equating coefficients of corresponding powers:

$x^2: \quad 1 = a + b$ ------ (1)

$x^1: \quad 5 = \quad -b + c$ ------ (2)

$x^0: -4 = a \quad -c$ ------ (3)

$(1)+(2): 6 = a + c$ ------ (4)

$(3)+(4): 2 = 2a$

$\therefore a = 1$

In (1): $1 = 1 + b$

$b = 0$

In (3): $-4 = 1 - c$

$c = 5$

(ii) $\int \frac{x^2+5x-4}{(x-1)(x^2+1)} dx = \int \left[\frac{1}{x-1} + \frac{5}{x^2+1}\right] dx$

$= \ln|x-1| + 5\tan^{-1} x + C$

(c)

$$I=\int\frac{dx}{x\sqrt{x^2-1}}$$ Let $x=\sec\theta$; $dx=\sec\theta\tan\theta d\theta$

$$I=\int\frac{\sec\theta\tan\theta d\theta}{\sec\theta\sqrt{\sec^2\theta-1}}$$
$$=\int\frac{\sec\theta\tan\theta d\theta}{\sec\theta|\tan\theta|}$$
$$=\int(\pm1)d\theta$$
$$=\pm\theta+C$$
$$=\pm\tan^{-1}\sqrt{x^2-1}+C$$

Alternatively

$$I=-\tan^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)+D$$

or

$$I=\cos^{-1}\left(\frac{1}{|x|}\right)+E$$

(d) (i)

RTP $\sqrt{\frac{4-x}{4+x}}=\frac{4-x}{\sqrt{16-x^2}}$

$$\text{LHS}=\sqrt{\frac{(4-x)(4-x)}{(4+x)(4-x)}}$$
$$=\sqrt{\frac{(4-x)^2}{16-x^2}}$$
$$=\frac{4-x}{\sqrt{16-x^2}}$$

= RHS (QED)

(ii)

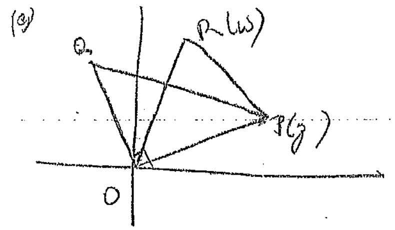
$$\int_{-2}^{2}\sqrt{\frac{4-x}{4+x}}dx=\int_{-2}^{2}\frac{4-x}{\sqrt{16-x^2}}dx$$
$$=\frac{1}{2}\int_{-2}^{2}\frac{8-2x}{\sqrt{16-x^2}}dx$$
$$=\frac{1}{2}\int_{-2}^{2}\frac{8}{\sqrt{16-x^2}}dx+\frac{1}{2}\int_{-2}^{2}\frac{-2x}{\sqrt{16-x^2}}dx$$
$$=\left[\frac{1}{2}\times8\times\sin^{-1}\left(\frac{x}{4}\right)\right]_{-2}^{2}+\left[\sqrt{16-x^2}\right]_{-2}^{2}$$
$$=4\left(\sin^{-1}\left(\frac{1}{2}\right)-\sin^{-1}\left(-\frac{1}{2}\right)\right)+\left(\sqrt{12}-\sqrt{12}\right)$$
$$=4\left(\frac{\pi}{6}-\left(-\frac{\pi}{6}\right)\right)$$
$$=\frac{4\pi}{3}$$

(e)

$$\int_0^1 2x\tan^{-1}x\,dx=\int_0^1\frac{d}{dx}(x^2)\tan^{-1}x\,dx$$
$$=\left[x^2\tan^{-1}x\right]_0^1-\int_0^1 x^2\frac{d}{dx}\tan^{-1}x\,dx$$
$$=\frac{\pi}{4}-\int_0^1\frac{x^2}{1+x^2}dx$$
$$=\frac{\pi}{4}-\int_0^1\frac{1+x^2-1}{1+x^2}dx$$
$$=\frac{\pi}{4}-\int_0^1 1dx+\int_0^1\frac{1}{1+x^2}dx$$
$$=\frac{\pi}{4}-[x]_0^1+\left[\tan^{-1}x\right]_0^1$$
$$=\frac{\pi}{4}-1+\frac{\pi}{4}$$
$$=\frac{\pi}{2}-1$$

2007 EXT 2 Q3

(a)



(i) Q is iz (1)

(ii) R is $z \operatorname{cis}\frac{\pi}{3}$

$\therefore$ QR is $z \operatorname{cis}\frac{\pi}{3} - iz = z\left(\operatorname{cis}\frac{\pi}{3} - i\right)$ (2)

(iii) $w^3 + z^3 = z^3 \operatorname{cis}\pi + z^3$

$= -z^3 + z^3$

$= 0$ (2)

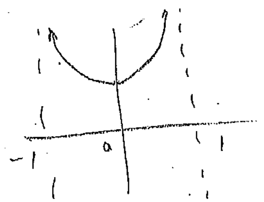
(b) (i) $y = x - \ln(\sec x + \tan x)$

$\frac{dy}{dx} = 1 - \frac{1}{\sec x + \tan x} \times (\sec x \tan x + \sec^2 x)$

$= 1 - \frac{1}{\sec x + \tan x} \times \sec x(\tan x + \sec x)$ (1)

$= 1 - \sec x$.

(ii)



$\sec x > 1$ $\quad 0 < x < \frac{\pi}{2}$

$\therefore -\sec x < -1$

$\therefore 1 - \sec x < 0$

$\therefore x - \ln(\sec x + \tan x)$ is decreasing.

When $x = 0$ $\quad \sec x + \tan x = 1$

$\therefore \ln(\sec x + \tan x) = 0$

$x - \ln(\sec x + \tan x) < 0$ for $0 < x < \frac{\pi}{2}$ (3)

(c) (i) As the polynomial has real coefficients, $\overline{2+i} = 2 - i$ is also a root. (1)

(ii) As the function is even, if x is a root, $-x$ is also a root. (2)

$\therefore -(2-i)$ is also a root.

(iii) Sum of roots $= (2+i) + (2-i) + (-2+i) + (-2-i) + \alpha + \beta$

$= \alpha + \beta$

$= 0$

$\therefore \alpha = -\beta$.

$(2+i)(2-i)(-2+i)(-2-i)\alpha\beta = -25$

$\therefore 25\alpha\beta = -25$

$\therefore \alpha\beta = -1$

$\therefore -\alpha^2 = -1$

$\therefore \alpha^2 = 1$

$\therefore \alpha = 1$ and $\beta = -1$. (3)

$\therefore$ Other roots are $-(2-i)$, 1, -1.

Solution to Question (4).

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

∵ (2,3) is on the ellipse.

$\therefore \frac{4}{a^2} + \frac{9}{b^2} = 1$

$\Rightarrow \frac{9}{b^2} = 1 - \frac{4}{a^2}$

$\therefore 9 = b^2\left(1 - \frac{4}{a^2}\right)$

$e = \frac{1}{2}, \quad e^2 = \frac{1}{4}$ — (1)

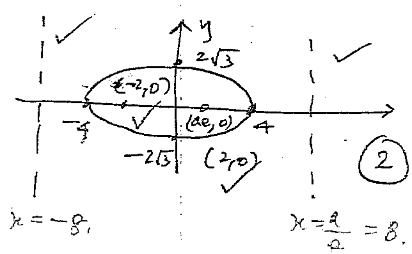
$b^2 = a^2(1 - e^2)$

$\therefore b^2 = \frac{3}{4}a^2$ — (2)

(2) $\therefore 9 = \frac{3}{4}a^2 - 3$

$\therefore a^2 = 16 \Rightarrow a = 4$

$b = \sqrt{12} = 2\sqrt{3}$.



(2)

(b) $\frac{x^2}{4} + \frac{y^2}{9} = 2$.

(ii) $\frac{x}{2} + \frac{2y}{9}\frac{dy}{dx} = 0$. (2)

$\frac{2\sqrt{2}\cos\theta}{2} + \frac{2 \cdot 3\sqrt{2}\sin\theta}{9}\frac{dy}{dx} = 0$

$\therefore \frac{2\sin\theta}{3}\frac{dy}{dx} = -\cos\theta$

$\Rightarrow dy/dx = \frac{-3\cos\theta}{2\sin\theta}$.

(i) $\frac{x^2}{4} + \frac{y^2}{9} = 2$

$\frac{(2\sqrt{2}\cos\theta)^2}{4} + \frac{(3\sqrt{2}\sin\theta)^2}{9}$

$= \frac{8\cos^2\theta}{4} + \frac{18\sin^2\theta}{9}$

$= 2(\cos^2\theta + \sin^2\theta) = 2$

∴ $P(2\sqrt{2}\cos\theta, 3\sqrt{2}\sin\theta)$ lie's on $\frac{x^2}{4} + \frac{y^2}{9} = 2$

∵ Coordinate satisfy the equation. (1)

(iii) $m' = \frac{2\sin\theta}{3\cos\theta}$ (2)

∴ equation of normal

$y - 3\sqrt{2}\sin\theta = \frac{2\sin\theta}{3\cos\theta}(x - 2\sqrt{2}\cos\theta)$ (3)

(iv) ∴ (3) passes through $(-2\sqrt{2}, 0)$

$-3\sqrt{2}\sin\theta = -\frac{4\sqrt{2}\sin\theta}{3\cos\theta}(1 + \cos$

$-9\cos\theta = -4 - 4\cos\theta$

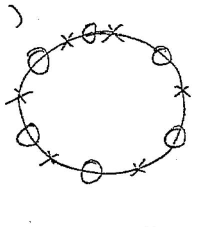
$\therefore \cos\theta = 0.8$ (2)

$\theta \doteqdot 37°$

0.646^c

(c)

(i)



$6! \times 5!$

720×120
86400

(2)

(ii) 2^6 (2)

QUESTION 5

(a) (i)

let $P(x) = (x-k)^2 \cdot Q(x)$ 2

$P'(x) = (x-k)^2 \cdot Q'(x) + 2(x-k)Q(x)$

$\therefore P'(x) = (x-k)\left[(x-k)Q'(x) + 2Q(x)\right]$

$P'(k) = 0 \quad [0 + 2Q(k)] = 0$

$\Rightarrow k$ is a zero of $P'(x)$.

(ii) 2

let $P(x) = x^{2n} - nx^{n+1} + nx^{n-1} + 1$

Now $P(1) = 1 - n + n + 1 = 0$

∴ $x = 1$ is a root of $P(x) = 0$

$P'(x) = 2nx^{2n-1} - n(n+1)x^n + n(n-1)x^{n-2}$

$P'(1) = 2n - n^2 - n + n^2 - n$

$= 0$

$\Rightarrow P(1) = P'(1) = 0$

∴ $x = 1$ is a double root of $P(x) = 0$

(b) inner radius of cyl. shell $4 - (x + \delta x)$

height $= y = \frac{x^2}{1+x^2}$ 2

$\therefore \delta V = 2\pi\left[4 - (x + \delta x)\right]\cdot\frac{x^2}{1+x^2}\cdot\delta x$

ie $\delta V = 2\pi\left[(4-x)\frac{x^2}{1+x^2}\right]\delta x - 2\pi\left[\frac{x^2}{1+x^2}\right](\delta x)^2$

$\therefore \delta V = 2\pi\left[(4-x)\frac{x^2}{1+x^2}\right]$ since term in $(\delta x)^2$ is neglected

$\Rightarrow \delta V = 2\pi\left[4 - x\right]\left[1 - \frac{1}{1+x^2}\right]$

Vol. solid $= \lim_{\delta x \to 0} \sum_{x=0}^{2} 2\pi(4-x)\left(1 - \frac{1}{1+x^2}\right)\delta x$

$V = 2\pi\int_0^2 (4-x)\left(1 - \frac{1}{1+x^2}\right)dx$

$V = 2\pi\int_0^2 (4-x)dx - 2\pi\int_0^2 \frac{4-x}{1+x^2}dx$

$= 2\pi\left[4x - \frac{x^2}{2}\right]_0^2 - 2\pi\left[4\tan^{-1}x - \frac{1}{2}\ln(1+x^2)\right]_0^2$

$= 2\pi[8 - 2] - 2\pi\left[4\tan^{-1}2 - \frac{1}{2}\ln 5\right]$

$= \left(12\pi - 8\pi\tan^{-1}2 + \pi\ln 5\right)$ units^3 3

(c) (i) $m\ddot{x} = -mg + R$ ↑

ie $m\ddot{x} = -gm - \frac{mv^2}{20}$ 2

$\ddot{x} = -9.8 - \frac{v^2}{20}$

$\ddot{x} = -\left(\frac{196 + v^2}{20}\right)$

(ii) $\ddot{x} = \frac{dv}{dt} = -\left(\frac{196+v^2}{20}\right)$ 4

$\int_{50}^{0}\frac{dv}{196+v^2} = -\frac{1}{20}\int_0^T dt$

$\frac{1}{14}\left[\tan^{-1}\frac{v}{14}\right]_{50}^{0} = -\frac{T}{20}$

$T = \frac{10}{7}\tan^{-1}\frac{25}{7}$

QUESTION 6

(i) $2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx}(-x+2y) = y - 2x$

$\frac{dy}{dx} = \frac{y-2x}{2y-x} = \frac{2x-y}{x-2y}$

(ii) Stat. pts $\frac{dy}{dx} = 0$ $\quad 2x = y$

Sub into eqn $x^2 - 2x^2 + 4x^2 = 3$

$x^2 = 1 \quad x = \pm 1$

$y = 2x$

Stat pts $(1, 2)$ $(-1, -2)$

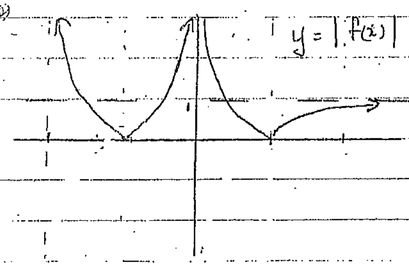
(iii) Vert. tangents when $\frac{dy}{dx}$ undefined.

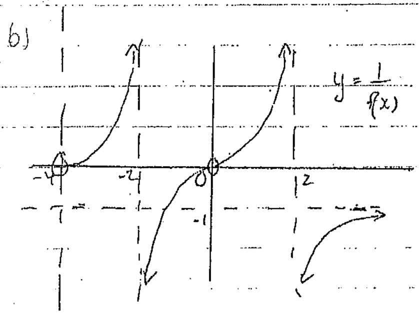
$x - 2y = 0$

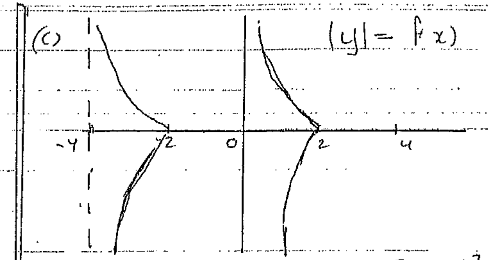
$x^2 - \frac{x^2}{2} + \frac{x^2}{4} = 3$

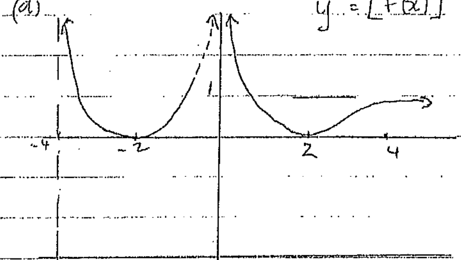
$\frac{3x^2}{4} = 3$

$x = \pm 1$

(iv) 

(b) 

(c) 

(d) 

(c)(i) $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \geq 0$

$x - 2 + \frac{1}{x} \geq 0$

$x + \frac{1}{x} \geq 2$

(ii) $\left(\frac{1}{a} + \frac{1}{b}\right)(a+b)$

$= 1 + \frac{a}{b} + \frac{b}{a} + 1$

$\frac{a}{b} + \frac{b}{a} \geq 2$ from (i)

$\therefore 2 + \frac{a}{b} + \frac{b}{a} \geq 4$

Least value $= 4$

(iii) $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

$= 1 + \frac{b}{a} + \frac{c}{a} + \frac{a}{b} + 1 + \frac{c}{b} + \frac{a}{c} + \frac{b}{c} + 1$

$= 3 + \frac{b}{a} + \frac{a}{b} + \frac{a}{c} + \frac{c}{a} + \frac{b}{c} + \frac{c}{b}$

$\geq 3 + 2 + 2 + 2$ from (i)

least value $= 9$.

Question 7

(a) $z^7 - 1 = 0$

(i) $1 + w + w^2 + w^3 + w^4 + w^5 + w^6$ is a GP where $a = 1$, $r = w$, $n = 7$.

$$S_7 = \frac{a(1-r^7)}{1-r} = \frac{1-w^7}{1-w} = \frac{1-1}{1-w} = 0$$

(ii) $(1+w)(1+w^2)(1+w^4) = (1+w^2+w+w^3)(1+w^4)$

$= 1 + w^2 + w + w^3 + w^4 + w^6 + w^5 + w^7$

$= (1 + w + w^2 + w^3 + w^4 + w^5 + w^6) + w^7$

$= 0 + 1$

$= 1$

(iii) One such equation is the monic quadratic

$\left(z - (w + w^2 + w^4)\right)\left(z - (w^6 + w^5 + w^3)\right) = 0$

Sum of roots $= -1$ (See Part (i))

Product of roots $= (w + w^2 + w^4)(w^6 + w^5 + w^3)$

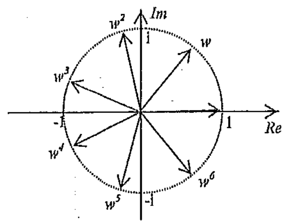
$= w^7 + w^6 + w^4 + w^8 + w^7 + w^5 + w^{10} + w^9 + w^7$

$= 1 + w^6 + w^4 + w + 1 + w^5 + w^3 + w^2 + 1$

$= 2 + (1 + w + w^2 + w^3 + w^4 + w^5 + w^6)$

$= 2$

Thus the equation is $z^2 + z + 2 = 0$

(iv) 

All angles $\frac{2\pi}{7}$

(b) (i) RTP: $(1+x)^n+(1-x)^n=2\sum_{k=0}^{n/2}{}^nC_{2k}x^{2k}$

$LHS=1+{}^nC_1x+{}^nC_2x^2+\dots+{}^nC_{n-1}x^{n-1}+x^n+1-{}^nC_1x+{}^nC_2x^2-\dots-{}^nC_{n-1}x^{n-1}+x^n$

$=2+2{}^nC_2x^2+2{}^nC_4x^4+\dots+2x^n$

$=RHS$ (QED)

(ii) (α) Two As can be arranged in ${}^5C_2$ ways. The remaining three letters can be chosen from 2 letters each, in 2^3 ways.

Hence the no. of ways overall is ${}^5C_2\times 2^3=80$.

(β) For 0 As: 2^n

For 2 As: ${}^nC_2\times 2^{n-2}$

For 4 As: ${}^nC_4\times 2^{n-4}$

For $n-2$ As: ${}^nC_{n-2}\times 2^2$

For n As: ${}^nC_n\times 2^0=1$

$\therefore$ Total

$=\sum_{k=0}^{n/2}{}^nC_{2k}2^{2k}$

$=\frac{1}{2}\left[(1+2)^n+(1-2)^n\right]$

$=\frac{1}{2}\left[3^n+(-1)^n\right]$

$=\frac{1}{2}(3^n+1)$ since n is even.

(c) (i) $x=ah+b$

When $h=0$, $x=3$ and when $h=10$, $x=9$, so

$3=b$

$9=10a+b$

$=10a+3$

$6=10a$

$a=\frac{3}{5}$

$\therefore x=\frac{3}{5}h+3$

$y=ch+d$

When $h=0$, $y=2$ and when $h=10$, $y=4$, so

$2=d$

$4=10c+d$

$=10c+2$

$2=10c$

$c=\frac{1}{5}$

$\therefore y=\frac{h}{5}+2$

(ii) $\delta V=xy\delta h$

Clearly $=\left(\frac{3}{5}h+3\right)\left(\frac{h}{5}+2\right)\delta h$

Thus

$$V=\int_0^{10}\left(\frac{3}{5}h+3\right)\left(\frac{h}{5}+2\right)dh$$

$$=\frac{1}{25}\int_0^{10}(3h+15)(h+10)\,dh$$

$$=\frac{1}{25}\int_0^{10}(3h^2+30h+15h+150)\,dh$$

$$=\frac{1}{25}\left[h^3+\frac{45h^2}{2}+150h\right]_0^{10}$$

$$=\frac{1}{25}\left[1000+\frac{4500}{2}+1500\right]$$

$=190$ unit^2

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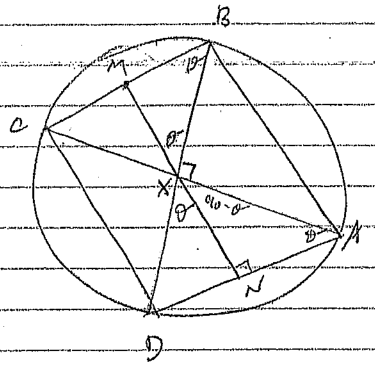
8 (a)



$\angle BXC = 90°$ (diagonals intersect at right angles)

$\therefore$ B, X and C lie on a circle with diameter BC

As M is midpoint of BC, M is the centre of the circle

$\therefore BM = MX$ (radii) (2)

$\therefore \angle MBX = \angle MXB$ (base angles of isosceles ΔBXM)

(ii) Let $\angle BXM = \theta$

$\angle BXM + \angle BXA + \angle AXN = 180°$ (straight angle)

$\therefore \theta + 90° + \angle AXN = 180°$

$\therefore \angle AXN = (90 - \theta)°$

$\angle XAN = \angle CAD = \angle CBD$ (angles at circumference $= \theta$ standing on same arc)

$\angle XAN + \angle ANX + \angle AXN = 180°$ (angle sum of Δ)

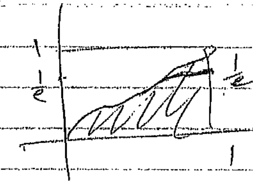
$\therefore \theta + \angle ANX + 90 - \theta = 180°$

$\therefore \angle ANX = 90°$

$\therefore MN \perp AD$ (3)

b (i) $I_n = \int_0^1 x^{2n+1} e^{-x^2} dx$

$0 \le x^{2n+1} e^{-x^2} \le 1$



$\therefore 0 \le \int_0^1 x^{2n+1} e^{-x^2} dx \le 1 \times 1$

$\therefore 0 \le \int_0^1 x^{2n+1} e^{-x} dx \le 1$ (1)

~~(ii) $I_n = \int_0^1 x^{2n+1} e^{-x^2} dx$ $\quad u = e^{-x^2}$, $v = \frac{1}{2n+2} x^{2n+2}$, $u' = -2xe^{-x^2}$, $v' = x^{2n+1}$~~

~~$= \left[e^{-x^2} \cdot \frac{1}{2n+2} x^{2n+2}\right]_0^1 - \int_0^1 \frac{1}{2n+2} x^{2n+2} (-2xe^{-x^2}) dx$~~

~~$= [$~~

(ii) $I_n = \int_0^1 x^{2n} \cdot x e^{-x^2} dx$ $\quad u = x^{2n}$, $v = -\frac{1}{2} e^{-x^2}$, $u' = 2n x^{2n-1}$, $v' = x e^{-x^2}$

$= \left[x^{2n} \cdot -\frac{1}{2} e^{-x^2}\right]_0^1 + n \int_0^1 x^{2n-1} e^{-x^2} dx$

$= \left[-\frac{1}{2} e^{-1} - 0\right] + n I_{n-1}$ (3)

$= -\frac{1}{2e} + n I_{n-1}$

(iii) $I_0 = \int_0^1 x e^{-x^2} dx$

$= \frac{1}{2}\left[-e^{-x^2}\right]_0^1$ (2)

$= -\frac{1}{2e} + \frac{1}{2}$

(iv) $S(0) \equiv 1 = e - \frac{2e I_0}{1!}$

$RHS = e - 2e\left(-\frac{1}{2e} + \frac{1}{2}\right)$

$= e + 1 - e$

$= 1$

Assume S(k) is true

i.e. $1+\frac{1}{1!}+\frac{1}{2!}+\dots+\frac{1}{k!} = e - \frac{2e I_k}{k!}$

Show S(k+1) is true

i.e. $1+\frac{1}{1!}+\frac{1}{2!}+\dots+\frac{1}{k!}+\frac{1}{(k+1)!} = e - \frac{2e I_{k+1}}{(k+1)!}$

LHS $= e - \frac{2e I_k}{k!} + \frac{1}{(k+1)!}$

$= e - \frac{1}{k!}\left(2e.\left(I_{k+1}+\frac{1}{2e}\right)\times\frac{1}{k+1}\right)+\frac{1}{(k+1)!}$

$= e - \frac{1}{(k+1)!}\left(2e I_{k+1}+1\right)+\frac{1}{(k+1)!}$

$= e - \frac{1}{(k+1)!}\left(2e I_{k+1}+1-1\right)$

$= e - \frac{2e I_{k+1}}{(k+1)!}$

= RHS.

∴ If S(k) is true, S(k+1) is true. (3)

S(1) is true and S(k+1) is true if S(k) is true

∴ By the process of Mathematical Induction S(n) is true for all integral $n \geq 0$

(v) $0 \le I_n \le 1$

∴ $0 \le \frac{I_n}{n!} \le \frac{1}{n!}$

∴ $-\frac{2e}{n!} \le -\frac{2e I_n}{n!} \le 0$

∴ $e-\frac{2e}{n!} \le e-\frac{2e I_n}{n!} \le e$

∴ $e-\frac{2e}{n!} \le 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots+\frac{1}{n!} \le e$

∴ $\lim_{n\to\infty} e-\frac{2e}{n!} \le \lim_{n\to\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+\dots+\frac{1}{n!}\right) \le e.$

∴ ~~$\lim_{n\to\infty}$~~ $e \le \lim_{n\to\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+\dots+\frac{1}{n!}\right) \le e.$

∴ $\lim_{n\to\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+\dots+\frac{1}{n!}\right) = e.$

∴ $1+\frac{1}{1!}+\frac{1}{2!}+\dots = e.$ (1)