SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2008

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 180 minutes.
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Start each **NEW** question in a separate answer booklet.

Total Marks - 120 Marks

- Attempt questions 1 - 8
- All questions are of equal value.

Examiner: *E. Choy*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n\neq -1;\ x\neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\ a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x>0$

Total marks – 120
Attempt Questions 1 - 8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find $\int_0^{\frac{\pi}{2}} \sin x \cos^4 x \, dx$ **2**

(b) Find $\int \frac{d\theta}{1-\cos\theta-\sin\theta}$ **3**

(c) Using the technique of integration by parts, evaluate $\int_1^e \frac{\log_e x}{x^2} dx$. **2**

(d) (i) Find real constants P and Q such that $\frac{7x-4}{2x^2-3x-2} = \frac{P}{2x+1} + \frac{Q}{x-2}$. **2**

(ii) Hence find $\int \frac{7x-4}{2x^2-3x-2} dx$ **2**

(e) Let $I_n = \int_0^1 \frac{x^n}{x^2+1} dx$, where n is an integer and $n \geq 0$.

(i) Show that $I_n + I_{n-2} = \frac{1}{n-1}$. **2**

(ii) Evaluate I_4. **2**

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Let $w = -5+7i$

(i) Sketch on a single Argand diagram w, $\overline{w}$, and $-w$. **2**

(ii) Find $\frac{1}{w}$ in the form of $x+iy$, where x and y are real numbers. **1**

(iii) On a separate Argand diagram, sketch the locus of $\arg(z-w) = \frac{\pi}{4}$. **2**

(iv) On a separate Argand diagram, sketch the locus of $\text{Re}(z-w) > 0$. **2**

(b) Let $z = \sqrt{3}+i$

(i) Express z in modulus-argument form. **1**

(ii) Show that $z^7 + 64z = 0$. **2**

(c) Suppose z_1, z_2 and z_3 are three complex numbers such that

$$\left.\begin{array}{r} |z_1| = |z_2| = |z_3| = 1 \\ z_1 + z_2 + z_3 = 0 \end{array}\right\}$$

Suppose also that z is a complex number, such that $|z| = 3$

(i) Show $|z-z_1|^2 = 10-(z\overline{z_1} + \overline{z}z_1)$. **3**

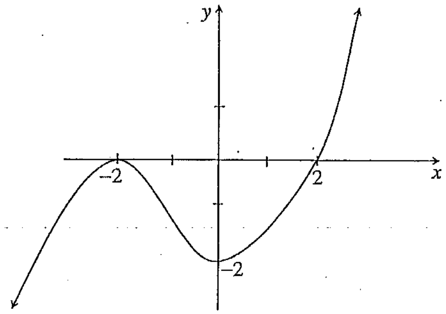
(ii) Show $|z-z_1|^2 + |z-z_2|^2 + |z-z_3|^2 = 30$. **2**

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Consider the relation defined by $2x^2+xy-y^2=0$.

Find the values of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $(2,4)$. **3**

(b) The graph below is that of $y=f(x)$



There is a maximum turning point at $(-2,0)$ and a minimum turning point at $(0,-2)$. The curve crosses the x-axis at $(2,0)$.

Sketch the following on separate diagrams, showing all essential features.

(i) $y=f(2x)$ **1**

(ii) $y=\dfrac{1}{f(x)}$ **2**

(iii) $y=\left[f(x)\right]^2$ **3**

(iv) $y^2=f(x)$ **3**

(v) $y=\tan^{-1}\left[f(x)\right]$ **3**

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) It is known that $2-i$ is a zero of the polynomial $p(x)$, where

$$p(x)=x^4-2x^3-x^2+2x+10$$

Express $p(x)$ as a product of real quadratic factors. **3**

(b) Given $p(x)=2x^3-3x^2-36x+2k$, where k is real. **3**

By considering turning points, prove that the equation $p(x)=0$ has three real and distinct roots if $-22<k<40\frac{1}{2}$.

(c) (i) Find all the complex roots of the equation $z^5-1=0$ in modulus and argument form. **2**

(ii) Show that $\dfrac{i\cos\theta-\sin\theta}{\cos\theta+i\sin\theta}=i$ for any angle θ. **2**

(iii) If $\dfrac{1+z}{1-z}=\cos\theta+i\sin\theta$, prove that $z=i\tan\dfrac{\theta}{2}$ **2**

(d) Using (c) (i) and (c) (iii), find all the complex roots of the equation $(1+z)^5=(1-z)^5$ in modulus and argument form.. **3**

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

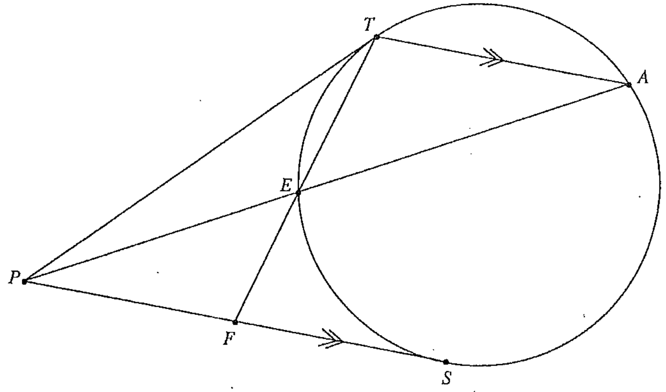
(a) Let α, β and γ be the roots of $x^3 - x^2 + 2x - 1 = 0$.

(i) Show that $\alpha + \beta = 1 - \gamma$ and hence find a polynomial equation with roots $-(\alpha+\beta)$, $-(\beta+\gamma)$ and $-(\gamma+\alpha)$. **3**

(ii) Find a polynomial equation with roots $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$. **3**

(iii) Evaluate $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. **2**

(b) The diagram below shows two tangents PT and PS drawn to a circle from a point P, exterior to the circle.



Through T, a chord TA is drawn parallel to the tangent PS. The secant PA meets the circle at E and TE produced meets PS at F.

(i) Prove $\Delta EFP \parallel\!\mid \Delta PFT$. **3**

(ii) Hence show that $PF^2 = TF \times EF$. **2**

(iii) Hence, or otherwise, prove that F is the midpoint of PS. **2**

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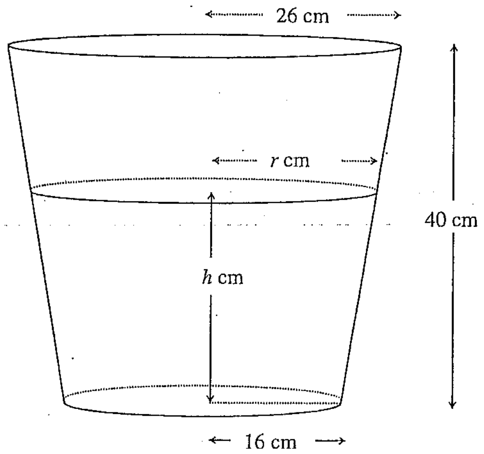
Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The flower pot from the home of a Mathematics Teacher is shown below. It has a circular base and top, and is cut from a cone.

The internal radius is 16 cm at the base and 26 cm at the top.
The depth of the pot is 40 cm.
The diagram below shows that any cross section parallel to the base is a circle.



(i) By considering a cross-sectional slice of radius r cm at h cm above the base, show that **2**

$$r = \frac{h}{4} + 16$$

(ii) Hence, find the volume of the flower pot. **3**

(b) (i) A wedge is cut from a right circular cylinder of radius r by two planes, one perpendicular to the axis of the cylinder, while the second makes an angle α with the first and intersects it at the centre of the cylinder. Find the volume of the wedge. **3**

(ii) The region under the curve $y = x(x-2)^2$ and between the x-intercepts is rotated about the y-axis. Find the volume of the solid by the cylindrical shells method. **2**

Question 6 continues on page 9

Question 6 (continued) **Marks**

(c) Given $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

(i) Prove that $f'(x) = \begin{cases} \frac{2}{1+x^2}, & |x| < 1 \\ -\frac{2}{1+x^2}, & |x| > 1 \end{cases}$ **2**

(ii) Discuss the behaviour of the function and its derivative at and around the point $x = 1$. **1**

(iii) Sketch the curve $y = f(x)$. **2**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Some years ago, there was correspondence in the Sydney Morning Herald about driving a shaft through the centre of the Earth and allowing a lift to fall freely through to the other side.

In one model of the motion, we treat the Earth as an object with all its mass concentrated at the centre. Then, Newton's Law of Gravitation gives the equation of motion as

$$v\frac{dv}{dr} = -\frac{GM}{r^2}$$

where r is the distance of the lift from the centre of the Earth, M is the mass of the Earth and G is a gravitational constant.

(i) If the lift falls from rest at the surface i.e. at $r = R$, show that the velocity of the lift is given by **3**

$$v = -\sqrt{2GM\left(\frac{1}{r} - \frac{1}{R}\right)}$$

(ii) Show that the time, T, taken for the lift to go from one side of the Earth to the other is given by **4**

$$T = \pi\sqrt{\frac{R^3}{2GM}}$$

(iii) Evaluate T, correct to 4 significant figures, if $G = 6\cdot6\times10^{-11}$ Nm^2/kg, $M = 5\cdot98\times10^{24}$ kg and $R = 6\cdot37\times10^{6}$ m **1**

(b) A second model for the motion of a lift falling through the centre of the Earth assumes that the density of the Earth is constant throughout.

Then, according to Newton's Law, the gravitational force at a distance r, from the centre of the earth, is due to the mass within that radius only, i.e. a fraction $\left(\frac{r}{R}\right)^3$ of the total mass, and the equation of motion is

$$v\frac{dv}{dr} = -\left(\frac{GM}{R^3}\right)r$$

(i) Show that the velocity of the lift at a distance r from the centre of the earth is given by **3**

$$v = -\sqrt{\left(\frac{GM}{R^3}\right)\left(R^2 - r^2\right)}$$

assuming that the lift falls from rest at the surface.

Question 7 continues on page 11

(b) (ii) By means of further integration, show that the time taken, from side to side is $\sqrt{2}T$, where T was defined in (ii) of (a) above i.e. $\pi\sqrt{\frac{R^3}{GM}}$. **4**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) How many arrangements of six zeros, five 1s, and four 2s are there in which:

(i) The first zero precedes the first 1 ? **2**

(ii) The first zero precedes the first 1 which precedes the first 2 ? **2**

(b) How many ways are there, on six successive nights, to invite one of three different friends over for dinner, so that no friend is invited more than three times? **3**

(c) Let m and n be two positive integers with $m > n$

(i) Show that $\sum_{r=1}^{n+1}\binom{2n+1}{n+r} = 2^{2n}$ **2**

(ii) By considering the coefficient of x^k in $(1+x)^m\left(1+\frac{1}{x}\right)^n$, show that **2**

$$\sum_{r=0}^{m-k}\binom{m}{k+r}\binom{n}{r} = \binom{m+n}{n+k}$$

where $m-n \le k \le m$.

(iii) A and B have $(n+1)$ and n fair coins respectively and they toss their coins simultaneously.

(α) Show that the probability that B gets r heads $(r = 0, 1, 2, ..., n)$ is given by $\frac{1}{2^n}\binom{n}{r}$. **1**

(β) Hence, find the probability that A will obtain k more heads than B, where $1 \le k \le n+1$. **2**

(γ) Show that the probability that A will obtain more heads than B is $\frac{1}{2}$. **1**

End of paper

2008 Trial HSC Mathematics Extension 2:
Solutions— Question 1

1\. (a) Find $\int_0^{\frac{\pi}{2}} \sin x \cos^4 x\, dx$ [2]

Solution: $$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} -\cos^4 x\, d\cos x, \\ &= \left[\frac{-\cos^5 x}{5}\right]_0^{\frac{\pi}{2}}, \\ &= \tfrac{1}{5}(-0+1), \\ &= \tfrac{1}{5}. \end{aligned}$$

(b) Find $\int \frac{d\theta}{1-\cos\theta-\sin\theta}$ [3]

Solution: $$\begin{aligned} \text{Put } t &= \tan\tfrac{\theta}{2}, \\ dt &= \frac{\sec^2\frac{\theta}{2}}{2}\,d\theta, \\ d\theta &= \frac{2\,dt}{1+t^2}. \\ I &= \int \frac{2\,dt}{(1+t^2)\left(1-\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}\right)}, \\ &= \int \frac{2\,dt}{1+t^2-1+t^2-2t}, \\ &= \int \frac{dt}{t(t-1)}. \\ \text{Now if } \frac{1}{t(t-1)} &= \frac{A}{t}+\frac{B}{t-1}, \\ 1 &= A(t-1)+Bt, \\ \text{put } t &= 0, \quad A=-1, \\ t &= 1, \quad B = 1. \\ \therefore I &= \int\left(\frac{1}{t-1}-\frac{1}{t}\right)dt, \\ &= \ln(t-1)-\ln t + c, \\ &= \ln(\tan^{-1}\tfrac{\theta}{2}-1)-\ln\tan^{-1}\tfrac{\theta}{2}+c. \end{aligned}$$

(c) Using the technique of integration by parts, evaluate $\int_1^e \frac{\log_e x}{x^2}\,dx$. [2]

Solution: $u = \ln x \qquad v' = \frac{dx}{x^2}$
$u' = \frac{1}{x}\,dx \qquad v = -\frac{1}{x}$

$$\begin{aligned} I &= -\frac{\ln x}{x}\Big]_1^e + \int_1^e \frac{dx}{x^2}, \\ &= -\frac{1}{e}+0-\left[\frac{1}{x}\right]_1^e, \\ &= -\frac{1}{e}-\left(\frac{1}{e}-1\right), \\ &= 1-\frac{2}{e}. \end{aligned}$$

(d) (i) Find real constants P and Q such that $\frac{7x-4}{2x^2-3x-2} = \frac{P}{2x+1}+\frac{Q}{x-2}$. [2]

Solution: $7x-4 = P(x-2)+Q(2x+1)$.
Put $x = 2$, $\quad 10 = 5Q$, $\quad \Rightarrow Q = 2$.
Put $x = -1/2$, $\quad -7^1/2 = -\frac{5P}{2}$, $\quad \Rightarrow P = 3$.

(ii) Hence find $\int \frac{7x-4}{2x^2-3x-2}\,dx$ [2]

Solution: $$\begin{aligned} I &= \int \frac{3\,dx}{2x+1}+\int\frac{2\,dx}{x-2}, \\ &= \tfrac{3}{2}\ln(2x+1)+2\ln(x-2)+c. \end{aligned}$$

(e) Let $I_n = \int_0^1 \frac{x^n}{x^2+1}\,dx$, where n is an integer and $n \geqslant 0$.

(i) Show that $I_n + I_{n-2} = \frac{1}{n-1}$. [2]

Solution: $$\begin{aligned} x^n &= x^{n-2}(x^2+1)-x^{n-2}. \\ \therefore I_n &= \int_0^1 x^{n-2}\,dx - \int_0^1 \frac{x^{n-2}}{x^2+1}\,dx, \\ &= \frac{x^{n-1}}{n-1}\Big]_0^1 - I_{n-2}, \\ I_n + I_{n-2} &= \frac{1}{n-1}. \end{aligned}$$

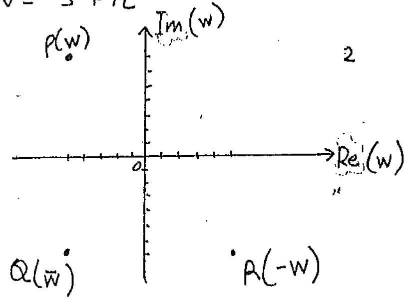
(ii) Evaluate I_4. [2]

Solution: $$\begin{aligned} I_4 + I_2 &= \frac{1}{3}. \\ I_2 &= \int_0^1 \frac{x^2+1-1}{x^2+1}\,dx, \\ &= [x-\tan^{-1}x]_0^1, \\ &= 1-\tfrac{\pi}{4}. \\ \therefore I_4 &= \tfrac{\pi}{4}-\tfrac{2}{3}. \end{aligned}$$

QUESTION 2

(a) $W = -5 + 7i$

(i)



2

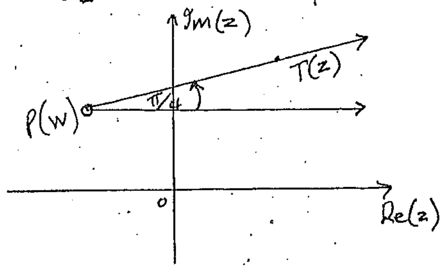
(ii) $\dfrac{1}{-5+7i} \times \dfrac{-5-7i}{-5-7i}$

$= \dfrac{-5-7i}{25+49}$ 1

$= \dfrac{-5}{74} - \dfrac{7i}{74}$

(iii) $\arg(z - w) = \frac{\pi}{4}$

ie $\arg[z - (-5+7i)] = \frac{\pi}{4}$ 2

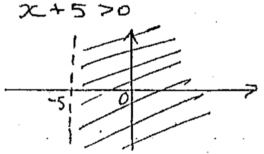


(iv) $Re(z - w) > 0$ let $z = x + iy$

$Re(x + iy + 5 - 7i) > 0$

ie $Re[(x+5) + i(y-7)] > 0$ 2

$\Rightarrow$ $x + 5 > 0$



(b) $z = \sqrt{3} + i$

(i) $|z| = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$

$\arg z = \frac{\pi}{6}$ 1

$\Rightarrow z = 2\,\text{cis}\frac{\pi}{6}$

(ii) $z^7 + 64z$

$= \left(2\,\text{cis}\frac{\pi}{6}\right)^7 + 64\left(2\,\text{cis}\frac{\pi}{6}\right)$

$= 2^7\,\text{cis}\frac{7\pi}{6} + 64(2)\,\text{cis}\frac{\pi}{6}$ 2

$= 128\left(\text{cis}\frac{7\pi}{6} + \text{cis}\frac{\pi}{6}\right)$

$= 128\left[-\frac{\sqrt{3}}{2} - \frac{i}{2} + \frac{\sqrt{3}}{2} + \frac{i}{2}\right]$

$= 0$

(c) (Many ways)

(i) Using $|z|^2 = z\bar{z}$ + conjugate props.

$|z - z_1|^2 = (z - z_1)(\overline{z - z_1})$ 3

$= (z - z_1)(\bar{z} - \bar{z}_1)$

$= z\bar{z} - z\bar{z}_1 - z_1\bar{z} + z_1\bar{z}_1$

$= |z|^2 - (z\bar{z}_1 + \bar{z}z_1) + |z_1|^2$

$= 9 - (z\bar{z}_1 + \bar{z}z_1) + 1$

$= 10 - (z\bar{z}_1 + \bar{z}z_1)$

(ii) $|z - z_1|^2 = 10 - (z\bar{z}_1 + \bar{z}z_1)$ using (i)

$|z - z_2|^2 = 10 - (z\bar{z}_2 + \bar{z}z_2)$

$|z - z_3|^2 = 10 - (z\bar{z}_3 + \bar{z}z_3)$ 2

Adding column wise $\Longrightarrow$

$|z - z_1|^2 + |z - z_2|^2 + |z - z_3|^2 =$

$30 - [z(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) + \bar{z}(z_1 + z_2 + z_3)]$

$= 30 - [z(\overline{z_1 + z_2 + z_3}) + \bar{z}(0)]$

$= 30 - [z(0) + \bar{z}(0)]$

$= 30$

Q3 (x2)

(a) given $2x^2 + xy - y^2 = 0$.

differentiate both sides with respect to x.

$4x + x\frac{dy}{dx} + y - 2y\frac{dy}{dx} = 0$ (A)

$\frac{dy}{dx}(2y - x) = 4x + y$

$\frac{dy}{dx} = \frac{4x + y}{2y - x}$

$\therefore$ at $(2,4)$ $\frac{dy}{dx} = \frac{8+4}{8-2} = \frac{12}{6} = \boxed{2.}$ ✓½

now $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{4x+y}{2y-x}\right)$

$= \dfrac{(2y-x)\left(4 + \frac{dy}{dx}\right) - (4x+y)\left(2\frac{dy}{dx} - 1\right)}{(2y-x)^2}$

$= \dfrac{(8-2)(4+2) - (8+4)(4-1)}{36.}$

$= \dfrac{36 - 36}{36.}$

$= \boxed{0}$ ✓½

OR From (A)

$4 + xy'' + y' + y' - [2yy'' + 2(y')^2] = 0$

$4 + 2y'' + 2 + 2 - [8y'' + 2 \times 4] = 0$

$8 + 2y'' - 8y'' - 8 = 0$

$-6y'' = 0$

$\boxed{y'' = 0.}$

OR. $2x^2 + xy - y^2 = 0$

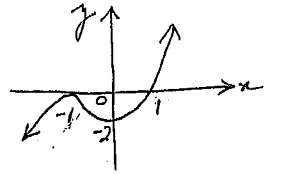
$\therefore (2x - y)(x + y) = 0$

$\therefore y = 2x$ or $y = -x$ (which is a pair of lines.)

now $(2,4)$ lies on $y = 2x$

$\therefore \dfrac{dy}{dx} = 2$

& $\dfrac{d^2y}{dx^2} = 0.$

(b) (i) 

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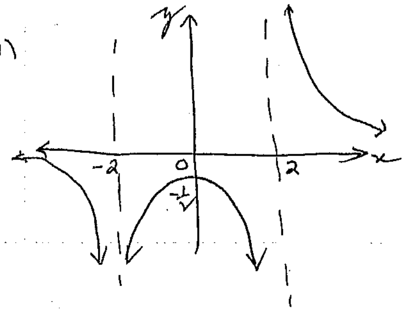
(ii) 

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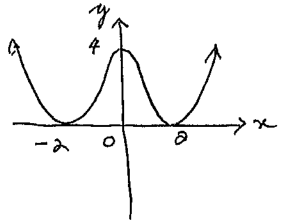
(iii) 

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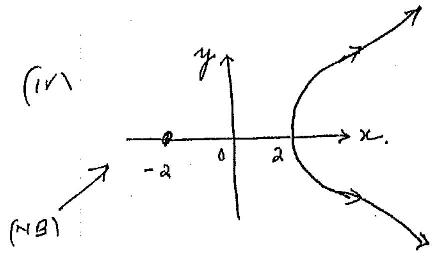
(iv) 

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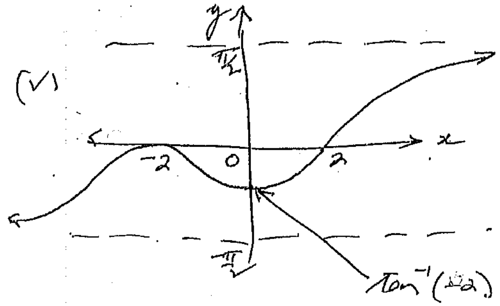
(v) 

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QUESTION 4.

(a) $[x-(2+i)][x-(2-i)]$

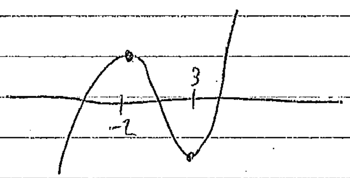
$= x^2 - [(2+i)-(2-i)]x + (2+i)\overline{(2+i)}$

$= x^2 - 4x + 5$

$$\begin{array}{r} x^2+2x+2 \\ x^2-4x+5 \,\big)\, \overline{x^4-2x^3-x^2+2x+10} \\ \underline{x^4-4x^3+5x^2} \\ 2x^3-6x^2+2x \\ \underline{2x^3-8x^2+10x} \\ 2x^2-8x+10 \\ \underline{2x^2-8x+10} \\ 0 \end{array}$$

$P(x) = (x^2-4x+5)(x^2+2x+2)$

(b)



$p'(x) = 6x^2 - 6x - 36$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3, -2$

$P(-2) > 0$ $\qquad$ $P(3) < 0$

$44 + 2k > 0$ $\qquad$ $-81 + 2k < 0$

$k > -22$ $\qquad$ $k < 40\frac{1}{2}$

$-22 < k < 40\frac{1}{2}$

C(i) $z^5 - 1 = 0$

$\text{cis}\,\frac{2\pi n}{5}$ when $n = 0, 1, 2, 3, 4$.

(ii) $\frac{i\cos\theta - \sin\theta}{\cos\theta + i\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$

$= \frac{i\cos^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + i\sin^2\theta}{\cos^2\theta + \sin^2\theta}$

$= i(\cos^2\theta + \sin^2\theta)$

$= i$

(iii) Let $t = \tan\frac{\theta}{2}$.

$\cos\theta + i\sin\theta = \frac{1-t^2}{1+t^2} + \frac{2ti}{1+t^2}$

$= \frac{1+2ti-t^2}{1+t^2}$

$= \frac{(1+it)^2}{(1+it)(1-it)}$

$= \frac{1+it}{1-it}$

So $\frac{1+it}{1-it} = \frac{1+z}{1-z}$

$\therefore z = i\tan\frac{\theta}{2}$.

(d) $(1+z)^5=(1-z)^5$

$$\left(\frac{1+z}{1-z}\right)^5-1=0.$$

Let $\omega=\frac{1+z}{1-z}$.

$\omega^5-1=0.$

$\omega=\text{cis}\frac{2\pi n}{5}$ where $n=0,1,2,3,4.$

$\therefore$ by c(iii). $z=i\tan\frac{\pi n}{5}$ where $n=0,1,2,3,4.$

Question 5

a) α,β and γ are roots of $x^3-x^2+2x-1=0$

i) $\alpha+\beta+\gamma=-\frac{b}{a}$

$\alpha+\beta+\gamma=1$

$\alpha+\beta=1-\gamma$

To find equation with roots $-(\alpha+\beta),-(\beta+\gamma),-(\gamma+\alpha)$

ie $\gamma-1,\ \alpha-1,\ \beta-1$

let $X=x-1$

$x=X+1$

$(X+1)^3-(X+1)^2+2(X+1)-1=0$

$X^3+3X^2+3X+1-X^2-2X-1+2X+2-1=0$

$\underline{\underline{X^3+2X^2+3X+1=0}}$

ii) To find equation with roots $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$

let $X=\frac{1}{x}$

$x=\frac{1}{X}$

$\left(\frac{1}{X}\right)^3-\left(\frac{1}{X}\right)^2+2\left(\frac{1}{X}\right)-1=0$ $\quad \times X^3$

$1-X+2X^2-X^3=0$

$\underline{\underline{X^3-2X^2+X-1=0}}$

iii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=-\frac{b}{a}$

$=\underline{\underline{2}}$

b) i) let $\angle PTF = x$

$\angle TAP = x$ (angle in alternate segment)

$\angle EPF = x$ (alt. $\angle$s, $PS \parallel TA$)

In Δ's EFP & PFT

$\angle EPF = \angle PTF = x$

$\angle TFP$ is common

$\therefore \Delta EFP \,|||\, \Delta PFT$ (equiangular)

ii) $\therefore \dfrac{PF}{FT} = \dfrac{EF}{PF}$ (corr. sides in same ratio)

$PF^2 = TF \times EF$

iii) $FS^2 = TF \times EF$ (square of tangent equals product of intercepts of secant)

$\therefore FS^2 = PF^2$

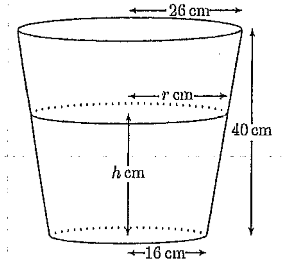
$FS = PF$

$\therefore F$ is midpoint of PS

2008 Trial HSC Mathematics Extension 2:
Solutions— Question 6

6. (a) The flower pot from the home of a Mathematics Teacher is shown below. It has a circular base and top and is cut from a cone.

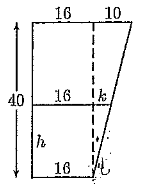
The internal radius is 16 cm at the base and 26 cm at the top.
The depth of the pot is 40 cm.
The diagram below shows that any cross-section parallel to the base is a circle.



(i) By considering a cross-sectional slice of radius r cm at h cm above the base, show that **[2]**

$$r = \frac{h}{4} + 16$$

Solution:



$$\frac{k}{h} = \frac{10}{40},$$
$$k = \frac{h}{4}.$$
$$\therefore\ r = 16 + \frac{h}{4}.$$

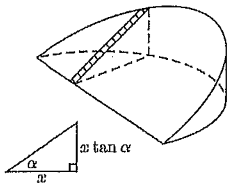
(ii) Hence find the volume of the flower pot. **[3]**

Solution:
$$\begin{aligned} V &= \pi \int_0^{40} \left(16 + \frac{h}{4}\right)^2 dh, \\ &= \frac{\pi}{16} \int_0^{40} (64 + h)^2\, dh, \\ &= \frac{\pi}{16} \left[\frac{(64+h)^3}{3}\right]_0^{40}, \\ &= \frac{\pi}{48}(1\,124\,864 - 262\,144), \\ &= \frac{53\,920\pi}{3}. \end{aligned}$$

i.e, The volume is 56 465 cm^3.

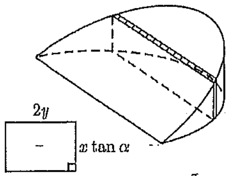
(b) (i) A wedge is cut from a right circular cylinder of radius r by two planes, one perpendicular to the axis of the cylinder, while the second makes an angle α with the first and intersects it at the centre of the cylinder. Find the volume of the wedge. $\boxed{3}$

Solution: Using triangular slices orthogonal to the y-axis—



$$\begin{aligned} \text{Area} &= \tfrac{1}{2} \times \sqrt{r^2 - y^2} \times \tan\alpha\sqrt{r^2 - y^2}. \\ \text{Volume} &= \frac{2\tan\alpha}{2} \int_0^r (r^2 - y^2)\, dy, \\ &= \tan\alpha \left[r^2 y - \frac{y^3}{3} \right]_0^r, \\ &= \tan\alpha \left\{ r^3 - \frac{r^3}{3} \right\}, \\ &= \tfrac{2}{3} r^3 \tan\alpha. \end{aligned}$$

Solution: Using rectangular slices orthogonal to the x-axis—

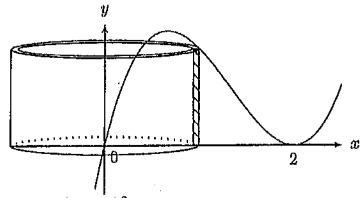


$$\begin{aligned} \text{Area} &= x\tan\alpha \times 2\sqrt{r^2 - x^2}. \\ \text{Volume} &= 2\tan\alpha \int_0^r x\sqrt{r^2 - x^2}\, dx. \\ \text{Put } x &= r\sin u, \\ dx &= r\cos u\, du, \\ \text{when } x &= 0, \quad u = 0, \\ x &= r, \quad u = \tfrac{\pi}{2}. \end{aligned}$$

$$\begin{aligned} \text{Volume} &= 2\tan\alpha \int_0^{\frac{\pi}{2}} r\sin u\sqrt{r^2 - r^2\sin^2 u}.r\cos u\, du, \\ &= 2r^3 \tan\alpha \int_0^{\frac{\pi}{2}} \cos^2 u \sin u\, du, \\ &= 2r^3\tan\alpha \left[\frac{-\cos^3 u}{3} \right]_0^{\frac{\pi}{2}}, \\ &= 2r^3 \tan\alpha \left\{ \tfrac{1}{3} - 0 \right\}, \\ &= \tfrac{2}{3} r^3 \tan\alpha. \end{aligned}$$

(ii) The region under the curve $y = x(x-2)^2$ and between the x-intercepts is rotated about the y-axis. Find the volume of the solid by the cylindrical shells method. $\boxed{2}$

Solution:



$$\begin{aligned} \text{Area} &= 2\pi x \times x(x-2)^2. \\ \text{Volume} &= 2\pi \int_0^2 (x^4 - 4x^3 + 4x^2)\, dx, \\ &= 2\pi \left[\frac{x^5}{5} - x_4 + \frac{4x^3}{3} \right]_0^2, \\ &= 2\pi \left\{ \frac{32}{5} - 16 + \frac{32}{3} - 0 \right\}, \\ &= \frac{32\pi}{15}. \end{aligned}$$

(c) Given $f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

(i) Prove that $f'(x) = \begin{cases} \dfrac{2}{1+x^2}, & |x| < 1 \\ \dfrac{-2}{1+x^2}, & |x| > 1 \end{cases}$ $\boxed{2}$

Solution: Put $y = sin^{-1}u$, $\quad u = \dfrac{2x}{1+x^2}$,

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}. \qquad \begin{aligned} \frac{du}{dx} &= \frac{(1+x^2).2 - 2x.2x}{(1+x^2)^2}, \\ &= \frac{2 + 2x^2 - 4x^2}{(1+x^2)^2}, \\ &= \frac{2 - 2x^2}{(1+x^2)^2}. \end{aligned}$$

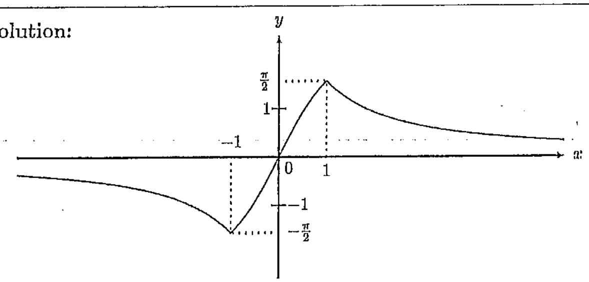
$$\begin{aligned} \text{Then } \frac{dy}{dx} &= \frac{1}{\sqrt{1 - \dfrac{4x^2}{(1+x^2)^2}}} \times \frac{2(1-x^2)}{(1+x^2)^2}, \\ &= \frac{2(1-x^2)}{\sqrt{1 + 2x^2 + x^4 - 4x^2}.(1+x^2)}, \\ &= \frac{2(1-x^2)}{|1-x^2|(1+x^2)}, \\ &= \begin{cases} \dfrac{2}{1+x^2}, & \text{if } |x| < 1, \qquad (\text{note } 1 - x^2 > 0) \\ \dfrac{-2}{1+x^2}, & \text{if } |x| > 1, \qquad (\text{note } 1 - x^2 < 0) \end{cases} \end{aligned}$$

(ii) Discuss the behaviour of the function and its derivative at and around the point $x = 1$ [1]

Solution: $\lim\limits_{x\to 1^-} \dfrac{dy}{dx} = 1, \qquad \lim\limits_{x\to 1^+} \dfrac{dy}{dx} = -1.$
$\therefore$ there is a cusp pointing upwards at $\left(1,\ \sin^{-1}(1)\right) = \left(1,\ \frac{\pi}{2}\right)$.
Also, as $f(x) = -f(-x)$, the function is odd.

(iii) Sketch the curve $y = f(x)$. [2]

Solution:



(a) (a) (i) $v\dfrac{dv}{dr} = -\dfrac{GM}{r^2}$

$\therefore \dfrac{d}{dr}\left(\frac{1}{2}v^2\right) = -GMr^{-2}$ $\frac{1}{2}$

$\therefore \frac{1}{2}v^2 = \dfrac{GM}{r} + c$ $\frac{1}{2}$

$r = R, v = 0 \Rightarrow c = -\dfrac{GM}{R}$ 1

3

$\therefore \frac{1}{2}v^2 = \dfrac{GM}{r} - \dfrac{GM}{R}$

$\therefore v^2 = 2GM\left(\dfrac{1}{r} - \dfrac{1}{R}\right)$ $\frac{1}{2}$

Taking the direction towards the centre of the Earth as negative then:

$\therefore v = -\sqrt{2GM\left(\dfrac{1}{r} - \dfrac{1}{R}\right)}$ $\frac{1}{2}$

(ii) $v = \dfrac{dr}{dt}$

$= -\sqrt{2GM\left(\dfrac{1}{r} - \dfrac{1}{R}\right)}$

$= -\sqrt{\dfrac{2GM}{R}\left(\dfrac{R-r}{r}\right)}$

4

$\therefore -\sqrt{\dfrac{2GM}{R}}\dfrac{dt}{dr} = -\sqrt{\dfrac{r}{R-r}}$ 1

$= -\dfrac{r}{\sqrt{r(R-r)}}$

$= \frac{1}{2}\left[\dfrac{(R-2r) - R}{\sqrt{Rr - r^2}}\right]$ $\frac{1}{2}$

$= \frac{1}{2}\left[\dfrac{(R-2r)}{\sqrt{Rr - r^2}} - \dfrac{R}{\sqrt{\left(\frac{R}{2}\right)^2 - \left(r - \frac{R}{2}\right)^2}}\right]$

$\therefore -\sqrt{\dfrac{2GM}{R}}\displaystyle\int_0^T \dfrac{dt}{dr}\,dr = \frac{1}{2}\int_R^{-R}\left[\dfrac{(R-2r)}{\sqrt{Rr - r^2}} - \dfrac{R}{\sqrt{\left(\frac{R}{2}\right)^2 - \left(r - \frac{R}{2}\right)^2}}\right]dr$ $\frac{1}{2}$

$\therefore -\sqrt{\dfrac{2GM}{R}}\displaystyle\int_0^T dt = \int_0^R\left[\dfrac{(R-2r)}{\sqrt{Rr - r^2}} - \dfrac{R}{\sqrt{\left(\frac{R}{2}\right)^2 - \left(r - \frac{R}{2}\right)^2}}\right]dr$

The time taken to go from the surface to the centre is double that of going from surface to surface.

$\therefore -\sqrt{\dfrac{2GM}{R}}T = \left[2\sqrt{Rr - r^2}\right]_0^R - R\left[\sin^{-1}\left(\dfrac{2r-R}{R}\right)\right]_0^R$ 1

$\therefore \sqrt{\dfrac{2GM}{R}}T = R\left[\sin^{-1}1 - \sin^{-1}(-1)\right] = R\pi$ $\frac{1}{2}$

$\therefore T = R\pi\sqrt{\dfrac{R}{2GM}} = \pi\sqrt{\dfrac{R^3}{2GM}}$ $\frac{1}{2}$

(iii) $T=\pi\sqrt{\dfrac{R^3}{2GM}}$

$=\pi\sqrt{\dfrac{\left(6\cdot 37\times 10^6\right)^3}{2\times 6\cdot 67300\times 10^{-11}\times 5\cdot 98\times 10^{24}}}$ (1)

$\approx 1\cdot 788\times 10^3$ seconds

$\approx 0\cdot 4966$ hours

$G=6\cdot 67300\times 10^{-11}\ \mathrm{m^3kg^{-1}s^{-2}}$

$G=6\cdot 6\times 10^{11}\ \mathrm{Mm^2/kg}$, $M=5\cdot 98\times 10^{24}$ kg and $R=6\cdot 37\times 10^6$ m

(b) (i)

$$v\frac{dv}{dr} = -\left(\frac{GM}{R^3}\right)r$$

$$\int_0^V v\,dv = -\left(\frac{GM}{R^3}\right)\int_R^r r\,dr$$

$$\frac{V^2}{2} = -\left(\frac{GM}{R^3}\right)\left[\frac{r^2}{2}\right]_R^r \quad (1)$$

$$\frac{V^2}{2} = -\left(\frac{GM}{R^3}\right)\left(\frac{r^2}{2}-\frac{R^2}{2}\right) \quad (1)$$

$$V = \pm\sqrt{\frac{GM}{R^3}\left(R^2-r^2\right)} \quad \left(\tfrac{1}{2}\right)$$

Since lift falls ↓ is negative $\left(\tfrac{1}{2}\right)$

$$\therefore V = -\sqrt{\left(\frac{GM}{R^3}\right)\left(R^2-r^2\right)}$$

(3)

(ii) $$\frac{dr}{dt} = -\sqrt{\frac{GM}{R^3}}\cdot\sqrt{R^2-r^2}$$

T_2 = time to reach other end

$$\int_R^{-R}\frac{1}{\sqrt{R^2-r^2}}\,dr = -\sqrt{\frac{GM}{R^3}}\int_0^{T_2}dt \quad 1$$

$$\left[\sin^{-1}\left(\frac{r}{R}\right)\right]_R^{-R} = -\sqrt{\frac{GM}{R^3}}\,T_2 \quad 1$$

$$\sin^{-1}(-1)-\sin^{-1}(1) = -\sqrt{\frac{GM}{R^3}}\,T_2$$

$$-\pi = -\sqrt{\frac{GM}{R^3}}\,T_2 \quad 1$$

$$\Rightarrow T_2 = \pi\sqrt{\frac{R^3}{GM}} = \pi\cdot\sqrt{2}\cdot\sqrt{\frac{R^3}{2GM}} = \sqrt{2}\,T \quad 1$$

(4)

Q8. Using cases.

a (i)

$$0\binom{14}{5}\binom{9}{5} + 20\binom{13}{5}\binom{8}{3} + 220\binom{12}{5}\binom{7}{2}$$

$$+ 2220\binom{11}{5}\binom{6}{1} + 22220\binom{10}{5}$$

$$= 252\,252 + 72072 + 16632 + 2772 + 252$$

$$= \boxed{343\,980}$$

OR Place the 4 2's ie $\binom{15}{4}$

Place the zero in first available slot.

The place remaining zeros ie $\binom{10}{5}$

$$\therefore \binom{15}{4} \times \binom{10}{5} = \boxed{343\,980.}$$

(ii) Using cases.

$01\ \binom{13}{5}\binom{8}{4}$

$001\ \binom{12}{4}\binom{8}{4}$

$0001\ \binom{11}{3}\binom{8}{4}$

$00001\ \binom{10}{2}\binom{8}{4}$

$000001\ \binom{9}{1}\binom{8}{4}$

$0000001\ \binom{8}{4}$

ie. $\boxed{140140.}$

OR Place zero in first position

$\therefore \binom{14}{5}$ for remaining zeros.

Place 1 in first remaining positions: then pick 4 places for remaining 1's.

$$\therefore \binom{14}{5}\binom{8}{4} = \boxed{140\,140.}$$

(b) The cases are.

$2,2,2.$ ie $\dfrac{6!}{2!2!2!} = 90.$

$3,2,1.$ ie. $\binom{3}{1} \times \binom{2}{1} \times \dfrac{6!}{3!2!} = 360$

$3,3.$ ie $\binom{3}{2} \times \dfrac{6!}{3!3!} = 3 \times \dfrac{6\times5\times4}{3\times2\times1.} = 60.$

Total $= 90 + 360 + 60 = \boxed{510}$

(C) Consider

(i) $$(1+x)^{2n+1} = \binom{2n+1}{0} + \binom{2n+1}{1}x + \binom{2n+2}{2}x^2 + \cdots$$

$$\cdots + \binom{2n+1}{r}x^r + \cdots + \binom{2n+1}{2n+1}x^{2n+1}.$$

Let $x=1$.

$$2^{2n+1} = \binom{2n+1}{0} + \binom{2n+1}{1} + \binom{2n+1}{2} + \binom{2n+1}{3} + \cdots$$

$$\cdots + \binom{2n+1}{r} + \cdots + \binom{2n+1}{2n+1}$$

now since $2n+1$ is odd there are $2n+2$ an even no. of terms.

$$\therefore 2^{2n+1} = 2\sum_{r=1}^{n+1}\binom{2n+1}{n+r} \quad \left[\underline{NB}\ \binom{2n+1}{r} = \binom{2n+1}{2n+1-r}\right]$$

$$\therefore \boxed{\sum_{r=1}^{n+1}\binom{2n+1}{n+r} = 2^{2n}}$$

(ii) now $$(1+x)^m\left(1+\frac{1}{x}\right)^n = \left[\binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots \cdots + \binom{m}{m}x^m\right]$$

$$\times\left[\binom{n}{0} + \binom{n}{1}\frac{1}{x} + \binom{n}{2}\cdot\frac{1}{x^2} + \cdots \cdots + \binom{n}{n}\frac{1}{x^n}\right]$$

$$= \frac{(1+x)^{m+n}}{x^n}$$

Consider the co-eff of x^k

$$LHS = \binom{m}{k}\binom{n}{0} + \binom{m}{k+1}\binom{n}{1} + \binom{m}{k+2}\binom{n}{2} + \cdots \cdots + \binom{m}{m}\binom{n}{m-k}$$

$$= \sum_{r=0}^{m-k}\binom{m}{k+r}\binom{n}{r}$$

now on the RHS. $$= \frac{1}{x^n}\left[\binom{m+n}{0} + \binom{m+n}{1}x + \binom{m+n}{2}x^2 + \cdots + \binom{m+n}{k+n}x^{k+n} + \cdots\right]$$

$$\therefore \text{Term containing } x^k = \frac{\binom{m+n}{k+n}x^{k+n}}{x^n}$$

$$\therefore \text{co-eff is } \binom{m+n}{k+n}$$

$$\therefore \sum_{r=0}^{m-k}\binom{m}{k+r}\binom{n}{r} = \binom{m+n}{k+n}.$$

(iii) (α) $P(X=r) = \binom{n}{r}\left(\frac{1}{2}\right)^{n-r}\cdot\left(\frac{1}{2}\right)^{r}$

$$= \binom{n}{r}\cdot\left(\frac{1}{2}\right)^{n-r+r}$$

$$= \binom{n}{r}\left(\frac{1}{2}\right)^{n}$$

(β) $$\sum_{r=0}^{n+1-k}\left[\frac{\binom{n+1}{k+r}}{2^{n+1}} \times \frac{\binom{n}{r}}{2^{n}}\right]$$

$$= \frac{1}{2^{2n+1}}\sum_{r=0}^{n+1-k}\binom{n+1}{k+r}\binom{n}{r}$$

$$= \frac{1}{2^{2n+1}}\binom{2n+1}{n+k}$$ (from c(ii) above)

(γ) $$\frac{1}{2^{2n+1}}\sum_{k=1}^{n+1}\binom{2n+1}{n+k} = \frac{1}{2^{2n+1}}\times 2^{2n}$$ (from c(i) above)

$$= \frac{1}{2}$$