SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2012

TRIAL HIGHER SCHOOL
CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 180 minutes
- Write using black pen.
- Board approved calculators may be used
- Show all necessary working in Questions 11–16
- A table of standard integrals is on the back of the multiple choice answer sheet

Total Marks - 100 Marks

Section I **10 Marks**

- Attempt Questions 1–10
- Allow about 15 minutes for this section.

Section II **90 Marks**

- Attempt Questions 11–16
- Allow about 2 hour 45 minutes for this section.

Examiner: *External Examiner*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

STUDENT NUMBER/NAME: ……………………..

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a>0, \ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x>0$

Section I **Objective-response Questions**
Total marks – 10
Attempt Questions 1 – 10

Answer each question on the multiple choice answer sheet provided.

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1 Let $u = 7\cos\frac{\pi}{4} + 7i\sin\frac{\pi}{4}$ and $v = a\cos b + ai\sin b$, where a and b are real constants.

If $uv = 42\cos\frac{\pi}{20} + 42i\sin\frac{\pi}{20}$, then

(A) $a = 35$ and $b = \frac{\pi}{5}$ (B) $a = 6$ and $b = \frac{\pi}{5}$

(C) $a = 35$ and $b = -\frac{\pi}{5}$ (D) $a = 6$ and $b = -\frac{\pi}{5}$

2 If $z^2 = 4\text{cis}\left(\frac{4\pi}{3}\right)$, then z is equal to

(A) $\sqrt{3}+i$ or $-\sqrt{3}-i$ (B) $1-\sqrt{3}i$ or $-1+\sqrt{3}i$

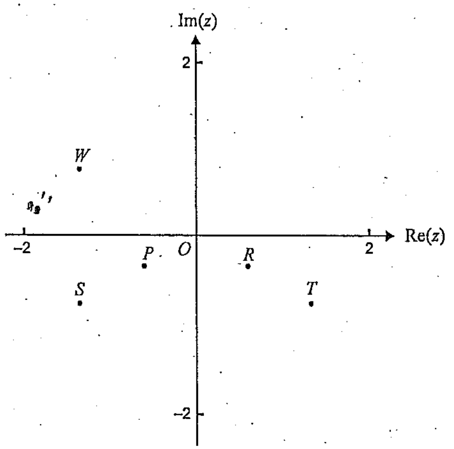
(C) $\sqrt{3}-i$ or $\sqrt{3}+i$ (D) $1-\sqrt{3}i$ or $1+\sqrt{3}i$

3 Let $z = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$.
The imaginary part of $z - i$ is

(A) $-\frac{i}{2}$ (B) $-\frac{3i}{2}$ (C) $-\frac{1}{2}$ (D) $-\frac{3}{2}$

4 The point W on the Argand diagram below represents a number w where $|w| = 1\cdot 5$.
The number w^{-1} is best represented by the point

(A) P (B) R (C) S (D) T



5 $P(z)$ is a polynomial in z of degree 4 with real coefficients
Which one of the following statements must be false?

(A) $P(z)$ has four real roots.
(B) $P(z)$ has two real roots and two non-real roots.
(C) $P(z)$ has one real root and three non-real roots.
(D) $P(z)$ has no real roots.

6 The graph of $f(x) = \frac{1}{x^2 + mx - n}$, where m and n are real constants, has no vertical asymptotes if

(A) $m^2 < -4n$ (B) $m^2 > -4n$ (C) $m^2 < 4n$ (D) $m^2 > 4n$

7 Consider the graph of $f(x) = \sin^3 x$ for $-\pi \le x \le 2\pi$.
The area bounded by the graph of $f(x)$ and the x-axis could be found by evaluating

(A) $\int_{-1}^{1}(1-u^2)\,du$ (B) $3\int_{-1}^{1}(1-u^2)\,du$

(C) $-\int_{-1}^{1}(1-u^2)\,du$ (D) $-3\int_{-1}^{1}(1-u^2)\,du$

8 Given that $\frac{dy}{dx} = y^2 + 1$, and that $y = 1$ at $x = 0$, then

(A) $y = y^2x + x + 1$ (B) $y = \tan\left(x + \frac{\pi}{4}\right)$

(C) $y = \tan\left(x - \frac{\pi}{4}\right)$ (D) $x = \log_e\left(\frac{y^2+1}{2}\right)$

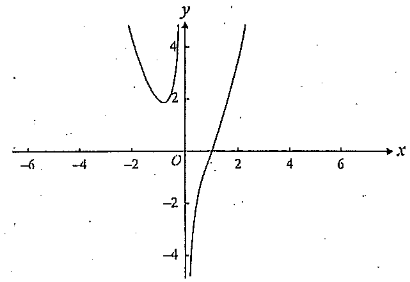
9 The velocity v m/s of a body which is moving in a straight line, when it is x m from the origin, is given by $v = \sin^{-1}x$.
The acceleration of the body in m/s^2 is given by

(A) $-\cos^{-1}x$ (B) $\cos^{-1}x$ (C) $-\frac{\sin^{-1}x}{\sqrt{1-x^2}}$ (D) $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$

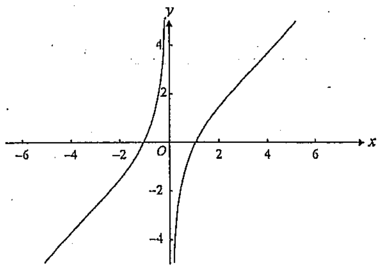
10 Let $f(x)=\dfrac{x^k+a}{x}$, where k and a are real constants.

If k is an odd integer which is greater than 1 and $a<0$, a possible graph of f could be

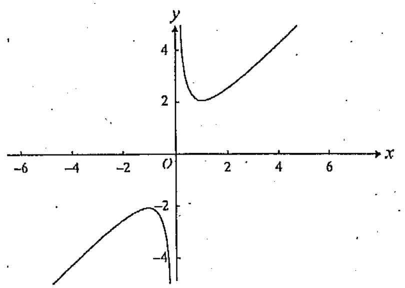
(A)



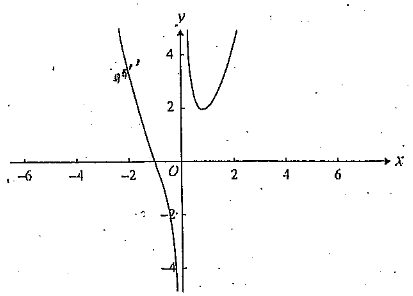
(B)



(C)



(D)



End of Section I

Section II **Free response Questions**

Total marks – 90

Attempt Questions 11 – 16

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Use the substitution $x=\sin^2\theta$ to evaluate $\displaystyle\int_0^{\frac{1}{2}}\frac{\sqrt{x}}{(1-x)^{\frac{3}{2}}}\,dx$ **3**

(b) Find $\displaystyle\int x\sqrt{3-x}\,dx$. **2**

(c) (i) By completing the square, find the exact value of $\displaystyle\int_{\frac{1}{8}}^{\frac{1}{4}}\frac{1}{\sqrt{2x(1-2x)}}\,dx$ **2**

(ii) Hence, evaluate $\displaystyle\int_{\frac{1}{8}}^{\frac{1}{4}}\frac{1-x}{\sqrt{2x(1-2x)}}\,dx$ **2**

(d) Find the value of the discriminant for the quadratic equation **2**

$$(1+i)z^2+4iz-2(1-i)=0$$

(e) (i) Find the value of $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^6$. **1**

(ii) Show that $(\cos\theta+i\sin\theta)(1+\cos\theta-i\sin\theta)=1+\cos\theta+i\sin\theta$. **1**

(iii) Hence show that $\left(1+\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^6+\left(1+\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^6=0$. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The line $x = 8$ is a directrix of the ellipse with equation **2**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad a>b>0$$

and (2, 0) is the corresponding focus.
Find the value of a and b.

(b) (i) Show that $2-i$ is a solution of the equation $z^3-(2-i)z^2+z-2+i=0$. **2**

(ii) Hence find all the solutions of the equation $z^3-(2-i)z^2+z-2+i=0$. **2**

(c) Consider the function $f(x)=\log_e\left(4-x^2\right)$.

(i) By first sketching $y=4-x^2$, sketch $y=f(x)$. **2**

Let A be the magnitude of the area enclosed by the graph of $y=f(x)$, the coordinate axes and the line $x=1$.

(ii) Without evaluating A, use (i) to show that $\log_e 3 < A < \log_e 4$. **1**

(iii) Find $\int \frac{x^2}{4-x^2}\,dx$. **3**

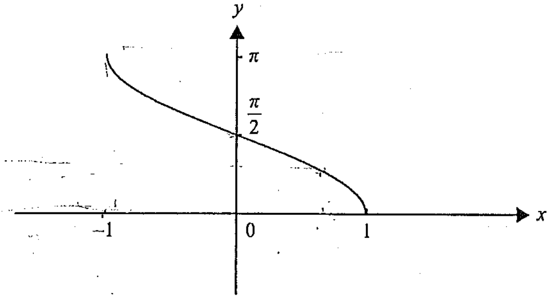
(iv) Hence find the exact value of A in the form $a+b\log_e c$, where a, b and c are integers. **3**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Prove using induction for integers $n \geq 2$. **3**

$$n^{n+1} > n(n+1)^{n-1}$$

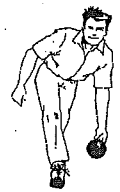
(b) The diagram below shows the graph of $y=\cos^{-1}x$.



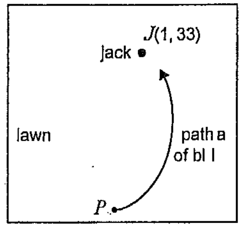
Using the method of cylindrical shells, find the exact volume formed if the graph above is rotated about the y-axis. **3**

Question 13 continues on the next page

(c) The game of lawn bowls is played on a horizontal lawn.
The aim is to roll a ball (usually called a 'bowl') to come to rest as close as possible to a target ball called the 'jack'.



Bowler



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All displacements are in metres.
At one stage during the game, the jack is at the point $J(1,33)$.
The path of a particular ball in this game is modelled by:

$$x = 2\sin\left(\frac{2t}{15}\right) \text{ and } y = 2 + \frac{5}{3}t - \frac{5}{3}\sin\left(\frac{t}{3}\right),\ 0 \le t \le \frac{15\pi}{2}$$

where t is the time in seconds after the ball is released from the point P.

(i) Write down the coordinates of P. **1**

(ii) Find expressions for the components of velocity, in metres per second, of the ball at time t seconds after the ball is released. **2**

(iii) At the instant the ball is released, what angle does its path make with the forward direction?
Give your answer correct to 1 decimal place. **2**

(iv) At what time, correct to the nearest tenth of a second, does the ball begin to swing left towards the jack? **2**

(v) Determine whether the path of the ball passes through J. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

A 'parasailing' water-skier i.e. a water-skier with a parachute attached of mass 90 kg is towed by a boat in a straight line from rest.
The boat exerts a constant force of 410 N acting horizontally on the skier.
At this stage the resistance acting on the skier is a constant 50 N, which acts horizontally.

(a) By use of a force diagram, show that the acceleration of the skier is 4 m/s^2. **2**

(b) By starting with $a = 4$, show that the speed of the skier, is given by $v^2 = 8x$, where x is the horizontal distance travelled by the skier.
Hence show that having been towed a distance of 32 m, his speed is 16 m/s. **2**

After the skier has been towed 32 m across the water the drag of the parachute becomes significant. The drag of the parachute produces an *additional* resistance of $6v$ N to the horizontal motion of the skier, where v m/s is the velocity of the skier.
Let a m/s^2 is the acceleration of the skier.

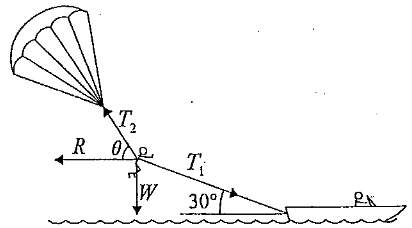
(c) Show that $a = \frac{1}{15}(60 - v)$ **1**

(d) Find the time required to reach a speed of 20 m/s from a speed of 16 m/s.
Give your answer in seconds, correct to one decimal place. **3**

After some time, the parasailing skier is being towed horizontally at a *constant speed* and at a fixed distance above the water.

The tow rope from the boat makes an angle of 30° to the horizontal, and the parachute cord makes an angle of θ to the horizontal.

The diagram below shows all the forces that are now acting on the parasailing water skier:
The tow rope now exerts a force, T_1, of 500 N on the skier.
The skier is experiencing a horizontal resistance, R, of 100 N.
Let the tension exerted by the parachute cord on the skier be T_2,
and the force due to gravity on the skier be W.
Take $g = 10$, where g is the magnitude of the acceleration due to gravity.



(e) By resolving in the horizontal and vertical directions, show that **3**

$$\begin{cases} 500\cos 30° - T_2\cos\theta - 100 = 0 \\ T_2\sin\theta - 500\sin 30° - 90g = 0 \end{cases}$$

Question 14 continues on the next page

Question 14 continued

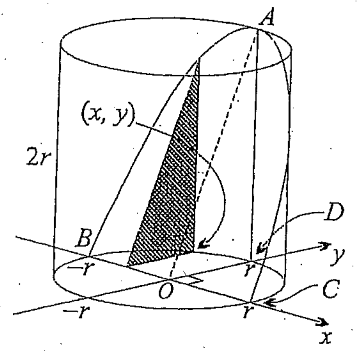
(f) Show that $\tan\theta = \dfrac{115}{25\sqrt{3}-10}$. **2**

(g) Hence, find the value of T_2 correct to the nearest integer. **2**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows a cylindrical wedge $ABCD$, the cross sections of which are all right triangles.
Each cross section is similar to triangle AOD.
The base of each cross section is parallel to OD.
The height of the cylinder is equal to the diameter of its base.
Let the radius of the base be r units.



(i) Show that the typical triangular cross-section shaded has area (r^2-x^2) square units. **2**

(ii) Hence find the volume of the wedge. **2**

(b) For positive real numbers x and y

(i) Prove that $\dfrac{x+y}{2} \geq \sqrt{xy}$ **2**
When is there equality?

(ii) Hence by considering $\dfrac{1}{a}+\dfrac{1}{b}$, or otherwise, prove that $\dfrac{2ab}{a+b} \leq \sqrt{ab}$ for positive real numbers a, b. **1**

(iii) Hence, or otherwise prove that $\dfrac{1}{x-1}+\dfrac{1}{x}+\dfrac{1}{x+1} > \dfrac{3}{x}$ for any $x>1$ **2**

(iv) If $H = 1+\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{4}+\dfrac{1}{5}+\dfrac{1}{6}+\dfrac{1}{7}+\ldots+\dfrac{1}{n}$, where n is an integer $n>1$, **2**
use (iii) to show that $\lim\limits_{n\to\infty} H = \infty$.

Question 15 continues on the next page

Question 15 continued

(c) (i) Given that ω is one of the non-real roots of $z^3 = 1$, show that $1+\omega+\omega^2 = 0$. **1**

(ii) Using (i), or otherwise, show that **3**

$$\left(\frac{\omega}{1+\omega}\right)^k + \left(\frac{\omega^2}{1+\omega^2}\right)^k = (-1)^k 2\cos\tfrac{2}{3}k\pi \text{, where } k \in \mathbb{Z}.$$

End of Question 15

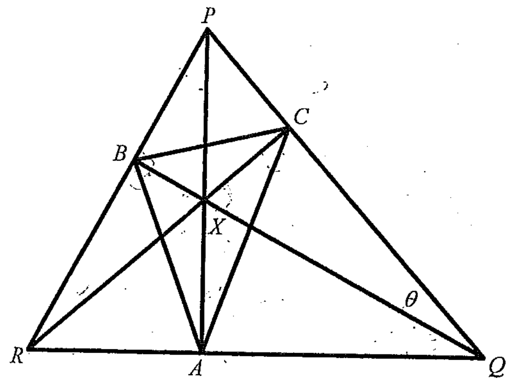
Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) $I_n = \int_0^a (a-x)^n \cos x \, dx$, $a > 0$ and n is an integer with $n \geq 0$.

(i) Show that, for $n \geq 2$, $I_n = na^{n-1} - n(n-1)I_{n-2}$. **3**

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)^3 \cos x \, dx$ **3**

(b) In the figure below, ΔPQR is acute angled and AP, BQ and CR are altitudes concurrent at X.
Also $\angle XQC = \theta$.
ΔABC is called the *pedal triangle* of ΔPQR.



(i) Prove that $\angle XRB = \theta$. **2**

(ii) Prove that X, A, Q and C are concyclic. **1**

(iii) Deduce that $\angle XAC = \theta$. **1**

(iv) Hence deduce that in an acute angled triangle the altitudes bisect the angles of the pedal triangle through which they pass. **2**

Question 16 continues on the next page

Question 16 continued

(c) (i) A binary string is a sequence of **1**s and **0**s,
e.g. **1 1 0 1 1 1 1 0 0 1 0 1** is a binary string of length 12.

In a binary string of length 50, how many ways are there to have a string with exactly 9 **1**s and that no two **1**s are adjacent? Justify your answer. **2**

(ii) Given 50 cards with the integers 1, 2, 3, … 50 printed on them, how many ways are there to select 9 distinct cards, such that no two cards have consecutive numbers printed on them?
(An answer with no reasoning will get no credit.) **1**

End of paper

Student Number:________________________

Mathematics Extension 2
Trial HSC 2012

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

correct

A ⊗ B ⊗ C ○ D ○

Section I: **Multiple choice answer sheet.**

Completely colour the cell representing your answer. Use black pen.

1. A B C **D**
2. A **B** C D
3. A B **C** D
4. **A** B C D
5. A B **C** D
6. **A** B C D
7. A **B** C D
8. A **B** C D
9. A B C **D**
10. **A** B C D

Q11 (a) $I = \int_0^{\frac{1}{2}} \frac{\sqrt{x}}{(1-x)^{3/2}}$ let $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

$= \int_0^{\frac{\pi}{4}} \frac{\sin\theta}{(1-\sin^2\theta)^{3/2}} \cdot 2\sin\theta\cos\theta\, d\theta$

$= 2\int_0^{\frac{\pi}{4}} \frac{\sin^2\theta\cos\theta\, d\theta}{\cos^3\theta}$

$= 2\int_0^{\frac{\pi}{4}} \tan^2\theta \cdot d\theta$

$= 2 \cdot \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1)\, d\theta$

$= 2\left[\tan\theta - \theta\right]_0^{\frac{\pi}{4}}$

$= 2\left[1 - \frac{\pi}{4}\right]$

$= \boxed{2 - \frac{\pi}{2}}$

(b) $\int x\sqrt{3-x}\, dx$. let $u = 3 - x$

$\therefore x = 3 - u$.

$dx = -du$.

$= -\int (3-u)\sqrt{u}\, du$

$= \int \left(u^{3/2} - 3u^{1/2}\right) du$

$= \frac{2}{5}u^{5/2} - 2u^{3/2} + c$

$= \boxed{\frac{2}{5}(3-x)^{5/2} - 2(3-x)^{3/2} + c}$

Q11 (CONTD)

c (i) $\displaystyle\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{1}{\sqrt{2x-4x^2}}\,dx.$

$= \frac{1}{2}\displaystyle\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{dx}{\sqrt{\frac{x}{2}-x^2}}.$

$= \frac{1}{2}\displaystyle\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{dx}{\sqrt{\frac{1}{16}-(x-\frac{1}{4})^2}}$

$= \frac{1}{2}\left[\sin^{-1}\frac{x-\frac{1}{4}}{\frac{1}{4}}\right]_{\frac{1}{8}}^{\frac{1}{4}}$

$= \frac{1}{2}\left[\sin^{-1}(4x-1)\right]_{\frac{1}{8}}^{\frac{1}{4}}$

$= \frac{1}{2}\left[\sin^{-1}0 - \sin^{-1}(-\frac{1}{2})\right]$

$= \frac{1}{2}\left(0 - -\frac{\pi}{6}\right)$

$= \frac{\pi}{12}.$

(ii) $\displaystyle\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{1-x}{\sqrt{2x-4x^2}}\,dx = \frac{1}{8}\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{8-8x}{\sqrt{2x-4x^2}}\,dx$

$= \frac{1}{8}\displaystyle\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{6+2-8x}{\sqrt{2x-4x^2}}\,dx.$

$= \frac{1}{8}\displaystyle\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{6}{\sqrt{2x-4x^2}}\,dx + \frac{1}{8}\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{2-8x}{\sqrt{2x-4x^2}}\,dx.$

$= \frac{3}{4}\displaystyle\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{dx}{\sqrt{2x-4x^2}} + \frac{1}{4}\left[\sqrt{2x-4x^2}\right]_{\frac{1}{8}}^{\frac{1}{4}}$

$= \frac{3}{4}\times\frac{\pi}{12} + \frac{1}{4}\left[\sqrt{\frac{1}{4}} - \sqrt{3/16}\right]$

$= \frac{\pi}{16} + \frac{1}{4}\left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right)$

$= \frac{\pi}{16} + \frac{1}{8} - \frac{\sqrt{3}}{16}.$

Q11 CONTD

(d) $\Delta = 16i^2 + 8(1-i^2)$

$= -16 + 8(2)$

$= 0.$

(e) (i) $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^6$

$= \cos\pi + i\sin\pi$

$= -1.$

(ii) $LHS = (\cos\theta + i\sin\theta)(1+\cos\theta - i\sin\theta)$

$= cis\,\theta\,(1 + cis(-\theta))$

$= cis\,\theta + cis\,0$

$= cis\,\theta + 1$

$= 1 + \cos\theta + i\sin\theta$

$= RHS.$

(iii) $\left(1+\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^6 + \left(1+\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^6$

$= \left[cis\frac{\pi}{6}\left(1+\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)\right]^6 + \left(1+\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^6$

$= -1\left(1+\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^6 + \left(1+\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^6$

$= 0$ as required.

2012 Extension 2 Mathematics THSC:
Solutions— Question 12

Marks

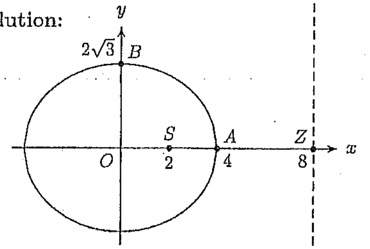
Question 12 (15 marks)

(a) The line $x = 8$ is a directrix of the ellipse with equation $\boxed{2}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0$$

and $(2,\,0)$ is the corresponding focus.
Find the value of a and b.

Solution:



$$\begin{aligned}
ae &= 2 \quad \ldots \quad \boxed{1}\\
\frac{a}{e} &= 8 \quad \ldots \quad \boxed{2}\\
\boxed{1} \times \boxed{2}: \; a^2 &= 16,\\
\therefore a &= 4.\\
\frac{SA}{AZ} &= e,\\
&= \tfrac{2}{4},\\
&= \tfrac{1}{2}.\\
b^2 &= a^2(1 - e^2),\\
&= 16(1 - \tfrac{1}{4}),\\
&= 12,\\
\therefore b &= 2\sqrt{3}.
\end{aligned}$$

(b) (i) Show that $2 - i$ is a solution of the equation $z^3 - (2-i)z^2 + z - 2 + i = 0$. $\boxed{2}$

Solution:

$$\begin{aligned}
\text{L.H.S.} &= (2-i)^3 - (2-i)(2-i)^2 + (2-i) - (2-i),\\
&= 0,\\
&= \text{R.H.S.}
\end{aligned}$$

$\therefore (2-i)$ is a solution.

(ii) Hence find all the solutions of the equation $z^3 - (2-i)z^2 + z - 2 + i = 0$. $\boxed{2}$

Solution:

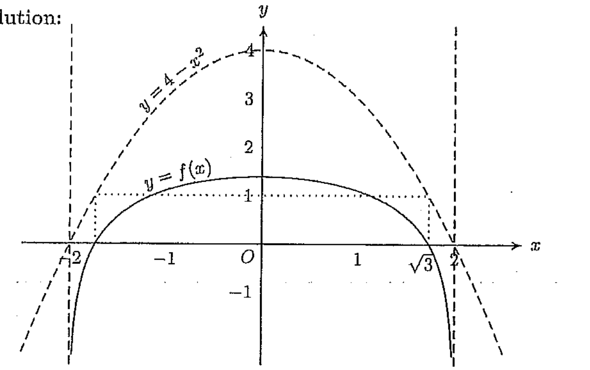
$$\begin{aligned}
z(z^2+1) - (2-i)(z^2+1) &= 0,\\
(z-2+i)(z^2+1) &= 0,\\
(z-2+i)(z+i)(z-i) &= 0.
\end{aligned}$$

$\therefore$ Solutions $2 - i$, $\pm i$.

(c) Consider the function $f(x) = \log_e(4 - x^2)$.

(i) By first sketching $y = 4 - x^2$, sketch $y = f(x)$. $\boxed{2}$

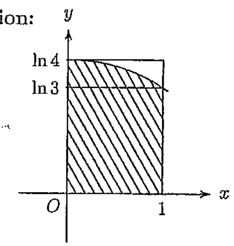
Solution:



Let A be the magnitude of the area enclosed by the graph of $y = f(x)$, the coordinate axes and the line $x = 1$.

(ii) Without evaluating A, use (i) to show that $\log_e 3 < A < \log_e 4$. $\boxed{1}$

Solution:



When $x = 0$, $f(0) = \ln 4$,
$x = 1$, $f(1) = \ln 3$.

From the sketch, it is clear that the shaded area A is between the upper rectangle's area $(1 \times \ln 4)$ and the lower rectangle's area $(1 \times \ln 3)$.

(iii) Find $\int \frac{x^2}{4-x^2}\,dx$. [3]

Solution: Method 1—

$$\begin{aligned}
I &= \int \frac{x^2-4+4}{4-x^2}\,dx, \\
&= -\int dx + \int \frac{4}{4-x^2}\,dx, \\
&= -x + \int \frac{dx}{2-x} + \int \frac{dx}{2+x}, \\
&= -x - \ln(2-x) + \ln(2+x) + c, \\
&= \ln\left(\frac{2+x}{2-x}\right) - x + c.
\end{aligned}$$

$$\begin{aligned}
\frac{4}{4-x^2} &\equiv \frac{A}{2-x} + \frac{B}{2+x}, \\
4 &\equiv A(2+x) + B(2-x), \\
\text{put } x = -2, &\quad B = 1, \\
x = 2, &\quad A = 1.
\end{aligned}$$

Solution: Method 2—

$$\begin{aligned}
I &= \int \frac{4\sin^2\theta . 2\cos\theta\, d\theta}{4-4\sin^2\theta}, \\
&= \int \frac{2\sin^2\theta \cancel{\cos\theta}}{\cos^{\cancel{2}}\theta}, \\
&= 2\int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta, \\
&= 2\int (\sec\theta - \cos\theta)\,d\theta, \\
&= 2\{\ln(\sec\theta + \tan\theta) - \sin\theta\} + c, \\
&= 2\ln\left(\frac{2}{\sqrt{4-x^2}} + \frac{x}{2}\cdot\frac{\cancel{2}}{\sqrt{4-x^2}}\right) - x + c, \\
&= 2\ln\left(\frac{x+2}{\sqrt{(2+x)(2-x)}}\right) - x + c, \\
&= \ln\left(\frac{2+x}{2-x}\right) - x + c.
\end{aligned}$$

$$\begin{aligned}
\text{Put } x &= 2\sin\theta, \\
dx &= 2\cos\theta\, d\theta, \\
\frac{x^2}{4} &= \sin^2\theta, \\
\frac{4-x^2}{4} &= \cos^2\theta.
\end{aligned}$$

(iv) Hence find the exact value of A in the form $a + b\log_e c$, where a, b and c are integers. [3]

Solution:

$$\begin{aligned}
I &= \int_0^1 \ln(4-x^2)\,dx, \\
&= x\ln(4-x^2)\Big]_0^1 + 2\int_0^1 \frac{x^2\,dx}{4-x^2}, \\
&= \{\ln 3 - 0\} + 2\left[\ln\left(\frac{2+x}{2-x}\right) - x\right]_0^1, \\
&= \ln 3 + 2\{\ln 3 - 1 - (\ln 1 - 0)\}, \\
&= 3\ln 3 - 2.
\end{aligned}$$

$$\begin{aligned}
u &= \ln(4-x^2), & v' &= 1, \\
u' &= \frac{-2x}{4-x^2}, & v &= x.
\end{aligned}$$

15 Ext 2

13. (a) Prove $n^{n+1} > n(n+1)^{n-1}$ for $n \geq 2$.

Let $n=2$. $LHS = 2^3 = 8$, $RHS = 2(3)^1 = 6$

$\therefore$ true for $n=2$.

Assume true for $n=k$

$k^{k+1} > k(k+1)^{k-1}$

or $k^k > (k+1)^{k-1}$ $*$

or $\frac{k^k}{(k+1)^{k-1}} > 1$.

Let $n=k+1$ RTP $(k+1)^{k+1} > (k+2)^k$

OR $\frac{(k+1)^{k+1}}{(k+2)^k} > 1$

Now $LHS = \frac{(k+1)^{k+1}}{(k+2)^k}$

$> \frac{(k+1)^{k+1}}{(k+2)^k} \times \frac{(k+1)^{k-1}}{k^k}$ since $\frac{(k+1)^{k-1}}{k^k} < 1$ from assumption

$> \frac{(k+1)^{2k}}{[k(k+2)]^k}$

$= \frac{[(k+1)^2]^k}{[k(k+2)]^k}$

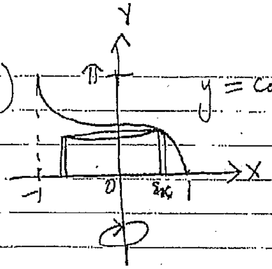
$= \left[\frac{k^2+2k+1}{k^2+2k}\right]^k > 1$ #

$\therefore$ true for $n=k+1$.

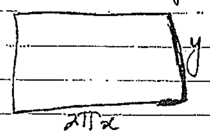
$\therefore$ By POMI, true $\forall n \geq 2$.

3

13 (b)



Area of cylindrical slice $= 2\pi xy$



$$\text{Vol. of cyl. slice} = 2\pi xy\,\delta x.$$

$$V = \lim_{\delta x \to 0} \sum_{x=-1}^{1} 2\pi xy\,\delta x.$$

$$= 2\pi \int_{-1}^{1} x\cos^{-1}x\,dx$$

$$V = 4\pi \int_0^1 x.\cos^{-1}x.dx \quad \text{(symmetric)} \qquad u = \cos^{-1}x \quad dv = x \qquad du = \frac{-1}{\sqrt{1-x^2}} \quad v = \frac{x^2}{2}$$

$$= 4\pi \left[\left[\frac{x^2}{2}.\cos^{-1}x\right]_0^1 - \int_0^1 \frac{x^2}{2}\left(\frac{-1}{\sqrt{1-x^2}}\right)dx \right]$$

$$= 2\pi \left[x^2 \cos^{-1}x\right]_0^1 + 2\pi \int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$$= 2\pi\left[x^2\cos^{-1}x\right]_0^1 + 2\pi\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta \qquad \text{Let } x = \sin\theta,\ dx = \cos\theta\,d\theta;\ x=0,\ \theta=0;\ x=1,\ \theta=\frac{\pi}{2}$$

$$= 2\pi\left[x^2\cos^{-1}x\right]_0^1 + 2\pi.\frac{1}{2}\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta$$

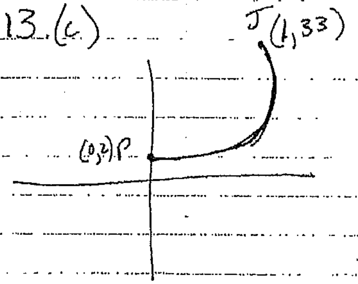
$$= 0 + \pi\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

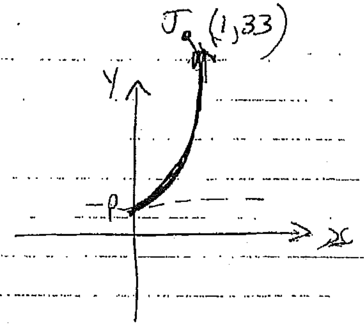
$$= \pi\left(\left(\frac{\pi}{2} - 0\right) - (0 - 0)\right)$$

$$= \frac{\pi^2}{2} \text{ cubic units.}$$

3

13.(c)





(i) $x = 2\sin\frac{2t}{15}$ $\quad y = 2 + \frac{5}{3}t - \frac{5}{3}\sin\left(\frac{t}{3}\right)$

$0 \le t \le \frac{15\pi}{2}$

At $t = 0$, $x = 0$, $y = 2$

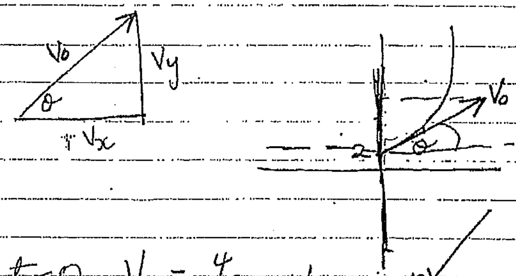
$\Rightarrow P = (0,2)$ 1

(ii) $V_x = 2\cos\frac{2t}{15} \times \frac{2}{15}$ and $V_y = \frac{5}{3} - \frac{5}{9}\cos\frac{t}{3}$ 2

$V_x = \frac{4}{15}\cos\frac{2t}{15}$ 1

2

(iii)



Velocity components

$V_x = V\cos\alpha$

$V_y = V\sin\alpha$

At $t=0$, $V_x = \frac{4}{15}$, $V_y = \frac{10}{9}$

Then $\tan\theta = \frac{V_y}{V_x}$

$= \frac{10}{9} \times \frac{15}{4}$

$\tan\theta = \frac{25}{6}$

$\theta = 1.33$ radians

or 76.5°

2

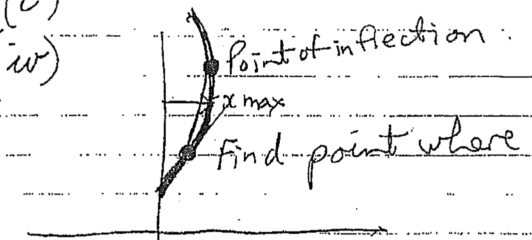
Then angle made with forward direction

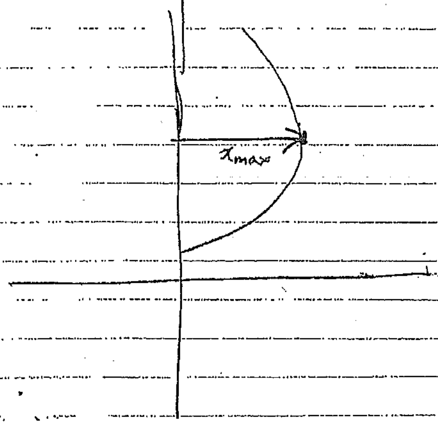
$= 90 - 76.5$

$= 13.5^\circ$

13(c)(iv)

*





$x = 2\sin\left(\frac{2t}{15}\right)$

$x' = \frac{4}{15}\cos\frac{2t}{15} = 0$ for stat pts

$\Rightarrow \cos\frac{2t}{15} = 0$ ✓ $\boxed{\frac{1}{2}}$

$\frac{2t}{15} = \frac{\pi}{2}, \frac{3\pi}{2}$ ✓ $\boxed{\frac{1}{2}}$

$t = \frac{15\pi}{4}, \frac{45\pi}{4}$ too big

$t = 11.78$ seconds

$\therefore$ at $t = 11.8$ seconds the ball starts to swing towards the jack. ✓ $\boxed{\frac{1}{2}}$

2

$0 < t < \frac{15\pi}{2}$

13(c)

(v) $x = 2\sin\frac{2t}{15}$, $y = 2 + \frac{5}{3}t - \frac{5}{3}\sin\left(\frac{t}{3}\right)$

Find t when $x = 1$, $y = 33$

$\Rightarrow 1 = 2\sin\frac{2t}{15}$ (1) and $33 = 2 + \frac{5}{3}t - \frac{5}{3}\sin\left(\frac{t}{3}\right)$ (2)

$31 = \frac{5}{3}t - \frac{5}{3}\sin\left(\frac{t}{3}\right)$

$\sin\frac{2t}{15} = \frac{1}{2}$

$\frac{2t}{15} = \frac{\pi}{6}, \frac{5\pi}{6}, \cancel{\frac{13\pi}{6}}$

$t = \frac{15\pi}{12}, \frac{75\pi}{12}, \cancel{\frac{195\pi}{12}}$ too big.

$t = \frac{5\pi}{4}, \frac{25\pi}{4}, \cancel{}$ ✓

Determine if values $t = \frac{15\pi}{12}$ or $\frac{75\pi}{12}$ satisfy (2).

$t = \frac{15\pi}{12} \Rightarrow 31 = \frac{5}{3} \times \frac{15\pi}{12} - \frac{5}{3}\sin\left(\frac{15\pi}{36}\right) = 4.935$ ✗

$t = \frac{75\pi}{12} \Rightarrow 31 = \frac{5}{3} \times \frac{75\pi}{12} - \frac{5}{3}\sin\left(\frac{75\pi}{12}\right)$

$31 = 31.546$ ✗ (close!)

$\therefore$ Ball does not hit the Jack. ✓

2

Question 14

(a) $O \longrightarrow x$

$R \longleftarrow \bullet \longrightarrow B$, $\rightarrow v$

$$\text{Nett Force} = B - R$$
$$= 410 - 50$$
$$= 360\,N$$

$F = ma$

$$\therefore 360 = 90a$$
$$a = 4\,m/s^2 \qquad [2]$$

(b) Given $a = 4$

$$\therefore \frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = 4$$

Integrate w.r.t. x:

$$\tfrac{1}{2}v^2 = 4x + C$$
$$v^2 = 8x + C'$$

Let $x = 0$ when $t = 0, v = 0$

$$\therefore C' = 0$$
$$\therefore v^2 = 8x$$

When $x = 32$

$$v^2 = 8 \times 32$$
$$= 256$$
$$\therefore v = 16\,m/s \quad (v > 0)$$

[2]

(c) $R \longleftarrow \bullet \longrightarrow B$, $P \longleftarrow$

$$P = 6v\,N$$

$$F = ma$$
$$410 - 50 - 6v = 90a$$
$$a = \frac{360 - 6v}{90}$$

[1] $$\therefore a = \frac{1}{15}(60 - v)$$

(d) $$\frac{dv}{dt} = \frac{1}{15}(60 - v)$$

$$\frac{dt}{dv} = \frac{15}{60 - v}$$

Time from $v = 16$ to $v = 20$

$$t = \int_{16}^{20} \frac{15}{60 - v}\,dv$$
$$= 15\left[-\ln(60 - v)\right]_{16}^{20}$$
$$= -15\left[\ln 40 - \ln 44\right]$$
$$\doteqdot 1.4\,s$$

[3]

Q14 (cont'd)

(e) **Horizontal** $a = 0$ $\quad \longrightarrow +$

Forces Boat: $500\cos 30^\circ$

Resistance: -100

Parachute: $-T_2\cos\theta$

$$\therefore 500\cos 30^\circ - T_2\cos\theta - 100 = 0 \quad -(1)$$

Vertical $a = 0$ $\quad \uparrow +$

Forces Boat: $-500\sin 30^\circ$

Weight: $-90g$

Parachute: $T_2\sin\theta$

$$\therefore T_2\sin\theta - 500\sin 30^\circ - 90g = 0 \quad -(2)$$

[3]

(f) from (1)

$$T_2\cos\theta = 500\cos 30^\circ - 100$$
$$= 250\sqrt{3} - 100$$

From (2)

$$T_2\sin\theta = 500\sin 30^\circ + 90g$$
$$= 500 \times \tfrac{1}{2} + 900$$
$$= 1150$$

$$\therefore \tan\theta = \frac{1150}{250\sqrt{3} - 100}$$
$$= \frac{115}{25\sqrt{3} - 10}$$

[2]

(g) $$T_1 = \frac{1150}{\sin\theta}$$

from (f) $\theta \doteqdot 73.85^\circ$

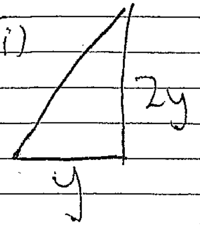
$$\therefore T_2 = 1197\,N$$

[2]

Q15

(a)

(i)



$$\text{Area} = \frac{1}{2} \times 2y \times y$$
$$= y^2.$$

Since $x^2 + y^2 = r^2$

$$\text{Area} = r^2 - x^2 \qquad 2$$

(ii)

$$\delta V = (r^2 - x^2)\,\delta x$$

$$V = \lim_{\delta x \to 0} \sum_{x=-r}^{r} (r^2 - x^2)\,\delta x.$$

$$= \int_{-r}^{r} r^2 - x^2 \, dx.$$

$$= 2\int_0^r r^2 - x^2 \, dx$$

$$= 2\left[xr^2 - \frac{x^3}{3}\right]_0^r$$

$$= 2\left(r^3 - \frac{r^3}{3}\right) \qquad 2$$

$$= \frac{4r^3}{3}.$$

(b) (i) $(a-b)^2 \geq 0$

$$a^2 - 2ab + b^2 \geq 0.$$

$$\frac{a^2+b^2}{2} \geq ab.$$

let $x = a^2 \Rightarrow a = \sqrt{x}$
$y = b^2 \Rightarrow b = \sqrt{y}.$

$$\frac{x+y}{2} \geq \sqrt{xy}. \qquad 2$$

(ii)

$$\frac{\frac{1}{a}+\frac{1}{b}}{2} \geq \frac{1}{\sqrt{ab}}$$

$$\frac{a+b}{2ab} \geq \frac{1}{\sqrt{ab}}. \qquad 1$$

$$\frac{2ab}{a+b} \leq \sqrt{ab}.$$

(iii)

$$\frac{x(x+1) + x^2 - 1 + x(x-1)}{x(x^2-1)} = \frac{3x^2-1}{x(x^2-1)}$$

$$= \frac{3x^2 - 3 + 2}{x(x^2-1)}$$

$$= \frac{3(x^2-1)}{x(x^2-1)} + \frac{2}{x(x^2-1)}$$

$= \frac{3}{x} + \frac{2}{x(x^2-1)}$

$x^2-1>0$ since $x>1$

$\therefore x(x^2-1)>0$

$\therefore \frac{2}{x(x^2-1)}>0$

So $\frac{1}{x-1}+\frac{1}{x}+\frac{1}{x+1} > \frac{3}{x}$

(iv) $H = 1 + \left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\left(\frac{1}{8}+\frac{1}{9}+\frac{1}{10}\right)+\dots+\left(\frac{1}{n-2}+\frac{1}{n-1}+\frac{1}{n}\right)$

$= 1+\frac{3}{3}+\frac{3}{6}+\frac{3}{9}+\dots+\frac{3}{n-1}$ let $n-1=3k$.

$= 1+1+\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\dots+\frac{1}{k}$.

$= 1+1+1+\dots+\frac{3}{k-1}$ let $k-1=3m$.

$= 1+1+1+\dots+\frac{1}{m}$.

As $n\to\infty$ this process can be continued

Thus $H = 1+1+1+1+\dots+1+\dots$

$\therefore \lim_{n\to\infty} H < \infty$

(c) (i) Since w is a solution of

$z^3=1$

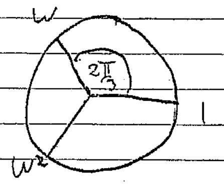
$\therefore w^3-1=0$.

$(w-1)(w^2+w+1)=0$

Since w is a non-real root $w\neq 1$.

$\therefore w^2+w+1=0$.

(ii)



$\arg(w) = \frac{2\pi}{3}$

Given that $1+w+w^2=0$

So $1+w=-w^2$

$1+w^2=-w$

LHS $= \left(\frac{w}{-w^2}\right)^k + \left(\frac{w^2}{-w}\right)^k$

$= (-1)^k(w^{-k}) + (-1)^k(w^k)$

$= (-1)^k(w^{-k}+w^k)$

Given $z^n+z^{-n}=2\cos n\theta$

$= (-1)^k\, 2\cos\left(\frac{2\pi}{3}k\right)$

Q16.

(a) (i) $I_n = \int_0^a (a-x)^n \cos x\,dx$ ---- (A)

$= \int_0^a (a-x)^n \frac{d}{dx}(\sin x)\,dx$

$= \left[(a-x)^n \sin x\right]_0^a - n\int_0^a (a-x)^{n-1}\sin x\,dx$

$= 0 - n\int_0^a (a-x)^{n-1}\sin x\,dx$

$= -n\int_0^a (a-x)^{n-1}\frac{d}{dx}(-\cos x)\,dx$

$= \left[-n\left((a-x)^{n-1}\cdot -\cos x\right)\right]_0^a + n(n-1)\int_0^a (a-x)^{n-2}\cdot -\cos x\,dx$

$= (0 + na^{n-1}) - n(n-1)\int_0^a (a-x)^{n-2}\cos x\,dx$

$\therefore \boxed{I_n = na^{n-1} - n(n-1)I_{n-2}}$ (B) $\boxed{\text{NB } n \geq 2}$

(ii) $I_3 = \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2}-x\right)^3\cos x\,dx$

$= 3\left(\frac{\pi}{2}\right)^2 - 6I_1$ — (C)

now $I_1 = \int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)\cos x\,dx$ from (A) [DON'T USE (B)]

$= \int_0^{\frac{\pi}{2}}\frac{\pi}{2}\cos x\,dx - \int_0^{\frac{\pi}{2}} x\cos x\,dx$

$= \left[\frac{\pi}{2}\sin x\right]_0^{\frac{\pi}{2}} - \left[\left[x\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\sin x\,dx\right]$

$= \frac{\pi}{2} - \left[\frac{\pi}{2} - 0 - \left[\cos x\right]_0^{\frac{\pi}{2}}\right]$

$= \frac{\pi}{2} - \left(\frac{\pi}{2} - 1\right)$

$= 1$

$\therefore \boxed{I_3 = \frac{3\pi^2}{4} - 6}$ from (C)

Q16. (CONTD)

(b) (i) BCQR is a cyclic quadrilateral

$\angle RCQ = \angle RBQ$ (90° altitudes)

ie angles are equal subtended by the chord RQ.

$\therefore \angle BRX = \angle BQC$ (angles in the same segment are equal)

ie $\angle XRB = \theta$.

(ii) $\angle XCQ = \angle XAQ = 90°$ (data)

$\therefore$ XAQ & C are concyclic (opposite angles supplementary)

(iii) $\angle XQC = \angle XAC = \theta$ (angles in the same segment standing on the same arc are equal)

(iv) BXAR is a cyclic quadrilateral

$\angle XBR = \angle XAR = 90°$ (data)

(opposite angles supplementary)

$\therefore \angle XRB = \angle XAB = \theta$ (angles in the same segment standing on same arc are equal)

$\therefore \angle XAB = \angle XAC = \theta$

$\therefore$ altitude PA bisects $\angle BAC$.

Similarly for the other angles.

Q16 (CONTD)

(C) (i) Place the 41 0's

choose the gaps and the end places

to places the 1's. (there are 42 spaces)

ie. $\binom{42}{9}$ ways to place the 1's.

(ii) Same logic as the above.

ie place the nine consecutive cards

into the 42 spaces such that no

two consecutive cards are adjacent

ie $\binom{42}{9}$.