SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

2012
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions:

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen. Black pen is preferred
- Board approved calculators may be used
- A table of standard integrals is provided on the back of the Multiple Choice answer sheet
- All necessary working should be shown in every question if full marks are to be awarded
- Marks may NOT be awarded for messy or badly arranged work
- Answer in simplest exact form unless otherwise stated

Total marks – 100 Marks

Section I Pages 2–4
10 marks

- Attempt Questions 1–10
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section

Section II Pages 5–10
90 marks

- Attempt Questions 11–16
- Allow about 2 hours 45 minutes for this section
- For Questions 11–16, start a new answer booklet per question

Examiner: Mr R. Boros

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0, \quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x>0$

Section I— 10 marks

Select the alternative A, B, C, or D that best answers the question.
Fill in the response oval on your multiple choice answer sheet.

Marks

1. $L + 2m - (L - 2n) - [2m + L - (2n - L)]$ simplifies to [1]
   (A) $4n + 2L$
   (B) $4n - 2L$
   (C) $4m + 4n - 2L$
   (D) $4m - 4n - 2L$

2. At 10% p.a. simple interest, how long will it take for a sum of money to double? [1]
   (A) 7.3 years
   (B) 5 years
   (C) 7.27 years
   (D) 10 years

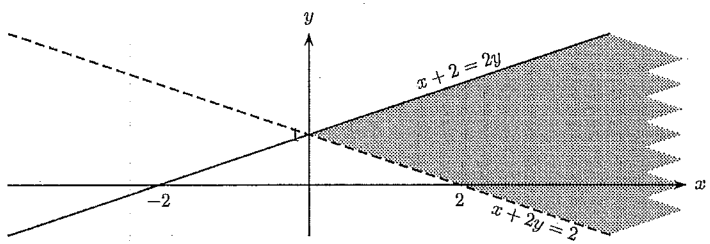
3. What is the value of k if the expression $4x^2 - 6x + k$ is a perfect square? [1]
   (A) $\frac{4}{9}$
   (B) $\frac{9}{4}$
   (C) 4
   (D) 9

4. $\frac{x^2 + 4x}{x^3 - 9x} \div \frac{x^2 + 2x - 8}{x^2 + x - 6}$ simplifies to [1]
   (A) 1
   (B) $\frac{x}{x - 3}$
   (C) $\frac{1}{x - 3}$
   (D) $\frac{1}{x + 3}$

5. The solution to the equation $2x^2 = 7x$ is $x =$ [1]
   (A) 0 or $-3\frac{1}{2}$
   (B) 0 or $3\frac{1}{2}$
   (C) $3\frac{1}{2}$ only
   (D) $3\frac{1}{2}$ or $-3\frac{1}{2}$

6. If p and q are the roots of $15x^2 + 75x - 3 = 0$ then $p + q =$ [1]
   (A) 75
   (B) 5
   (C) $-\frac{1}{5}$
   (D) -5

7. [1]



The shaded region in the diagram satisfies
   (A) $x + 2 \geqslant 2y$ and $x + 2y > 2$
   (B) $x + 2 \geqslant 2y$ and $x + 2y < 2$
   (C) $x + 2 \leqslant 2y$ and $x + 2y > 2$
   (D) $x + 2 \leqslant 2y$ and $x + 2y < 2$

8. $\log_3 15 + \log_3 18 - \log_3 10 =$ [1]
   (A) 1
   (B) 2
   (C) 3
   (D) 0

9. Two cards are drawn in succession from a regular pack of 52 cards. What is the probability that both cards are diamonds or both cards are clubs? [1]

(A) $\frac{2}{17}$
(B) $\frac{3}{5}$
(C) $\frac{3}{17}$
(D) $\frac{27}{52}$

10. If the 5th term and 18th term of an arithmetic series are 12 and 64 respectively, find the common difference. [1]

(A) -5
(B) 4
(C) -4
(D) 5

Section II— 90 marks

Marks

Question 11 (15 marks) (use a separate answer booklet)

(a) Find the first derivative of

(i) $y = (x^2 - 1)^3$, [2]

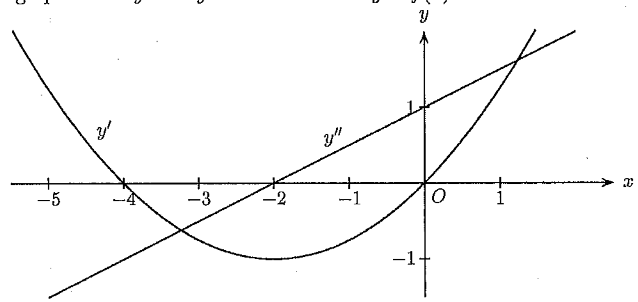
(ii) $y = \dfrac{2x}{x-1}$, [2]

(iii) $f(x) = \ln(3 - x)$. [1]

(b) Evaluate $\displaystyle\int_3^8 \sqrt{x+1}\,dx$. [3]

(c) Find the equation of the normal to the curve $y = \tan x$ at the point where $x = \frac{\pi}{4}$ (answer in the general form of a line). [3]

(d) The graph shows y' and y'' for the function $y = f(x)$. [3]



Sketch a graph of $y = f(x)$, clearly showing the x values of any turning points and points of inflexion.

(e) Find $\displaystyle\int 3\cos\left(\frac{x}{2}\right)\,dx$. [1]

Marks

Question 12 (15 marks) (use a separate answer booklet)

(a) $A(-1, 8)$, $B(4, -2)$, and $C(-3, -1)$ are three points on the number plane. The line ℓ_1 passes through the points A and B.

(i) Draw a sketch showing A, B, C, and ℓ_1. **1**

(ii) Find the exact distance AB. **1**

(iii) Show that ℓ_1 has the equation $2x + y - 6 = 0$. **1**

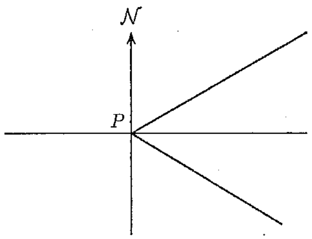
(iv) Find the perpendicular distance from the point C to the line $2x+y-6=0$. **2**

(v) Calculate the area of the triangle ABC. **2**

(vi) Find the co-ordinates of the midpoint, M, of AC. **1**

(vii) Find the equation of the line, ℓ_2, through M and parallel to AB (written in the general form of a line). **2**

(b) Two separate 'one man' canoes start off from a jetty, P, on a very large lake. The first canoeist paddles on a bearing of 040° T for 12 nautical miles to a buoy Q. At the same time the second canoeist paddles a distance of 8 nautical miles on a bearing of 100° T to another buoy R.



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(i) Copy the sketch above and add all the relevant information. **1**

(ii) Calculate the distance (in nautical miles) between the canoeists correct to one decimal place. **2**

(iii) If the two canoeists conduct a quick search of $\triangle PQR$ for any other canoeists, calculate the total area searched, giving your answer in square kilometres correct to the nearest 10 square kilometres. (Note: 1 nautical mile = 1852 metres.) **2**

Marks

Question 13 (15 marks) (use a separate answer booklet)

(a) (i) For the curve $y = 3\sin 4x$ in the domain $0 \leqslant x \leqslant \pi$, state the

(α) period, **1**

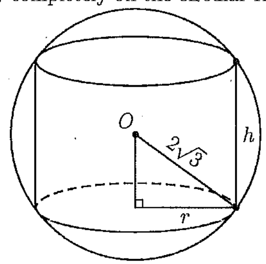
(β) amplitude. **1**

(ii) Sketch the curve $y = 3\sin 4x$, $0 \leqslant x \leqslant \pi$, clearly showing where the curve cuts the x-axis. **2**

(iii) Hence or otherwise, find the NUMBER of solutions to $\sin 4x = \cos x$ where $0 \leqslant x \leqslant \pi$. **2**

(b) Evaluate $\displaystyle\int_0^1 2xe^{(3x^2-5)}\,dx$, giving your answer to 3 significant figures. **3**

(c) A right circular cylinder of radius r cm and height h cm has to be designed to fit inside a sphere of $2\sqrt{3}$ cm radius so that both the bottom and the top touch the sphere (centre O) completely on the circular rim.



(i) Using the diagram as a guide, show that $r^2 = 12 - \dfrac{h^2}{4}$. **1**

(ii) If the volume of the cylinder is V, show that $V = 12\pi h - \frac{1}{4}\pi h^3$. **1**

(iii) Hence find the dimensions of the cylinder to give maximum volume. **2**

(d) Evaluate $\displaystyle\int_0^5 \frac{x}{5+3x^2}\,dx$, leaving your answer in exact form. **2**

Marks

Question 14 (15 marks) (use a separate answer booklet)

(a) Given that $x^2 - 6x - 7 = 8y$, find:

(i) the co-ordinates of the vertex, [1]

(ii) the co-ordinates of the focus, [1]

(iii) the equation of the directrix. [1]

(b) A road grader removes $V\,\text{m}^3$ of soil in t minutes, where $V = 25t - \dfrac{t^2}{50}$. [2]
Find the rate at which the soil is being removed after five minutes.

(c) A driver in a car is at a point A, from which branches out two roads. If he takes the road on the LEFT and journeys some distance, this road leads to a point B from which branches off three roads, one of which leads to his destination C. However if he takes the road on the RIGHT, and journeys along a certain distance, this road leads to a point D, from which branches off four roads, one of which leads to his destination C. [3]
Assuming he has no Sat. Navigation or prior knowledge of any of theses facts, except that he wants to travel to destination C, find the probability that he DOES NOT reach C on his first try.

(d) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\,dx + \int_0^{\frac{\pi}{2}} \cos^2 x\,dx$. [2]

(e) Let $A(4, 0)$ and $B(1, 0)$ be two fixed points and let P be the variable point (x, y).

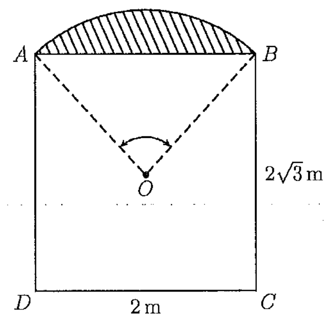
(i) Write down expressions for the distances PA and PB in terms of x and y. [2]

(ii) Find the locus of P whose distance from A is twice its distance from B. [3]

Marks

Question 15 (15 marks) (use a separate answer booklet)

(a) The diagram below shows an ancient window which consists of a rectangle $ABCD$ with height $2\sqrt{3}$ m and width 2 m surmounted by a minor segment of a circle which is stained glass. The centre of the circle is at O, the point of intersection of the diagonals of the rectangle.



(i) Explain why $A\hat{O}B = 60°$. [2]

(ii) Find the area of the minor segment correct to 3 decimal places. [2]

(b) (i) Sketch the region beneath the curve $y = e^x + 1$ which is above the x-axis and between the lines $x = 0$ and $x = 1$. [1]

(ii) The region in (b)(i) is now rotated about the x-axis. Find the volume of the resulting solid of revolution. Leave your answer in exact form. [3]

(c) For the curve $y = xe^{-x}$,

(i) *Prove* that $\dfrac{dy}{dx} = -e^{-x}(x-1)$. [1]

(ii) Find any stationary points and determine their nature. [2]

(iii) *Prove* that $\dfrac{d^2y}{dx^2} = e^{-x}(x-2)$. [1]

(iv) Show that there is a point of inflexion on this curve and find the co-ordinates of this point. [2]

(v) Sketch the curve, showing the co-ordinates of the point of inflexion and any stationary points. [1]

Marks

Question 16 (15 marks) (use a separate answer booklet)

(a) The number of DVD copies sold at a store of *The London Olympics 2012 Opening Ceremony* has increased exponentially in accordance with the formula $N = Ae^{kt}$ where t is the time in weeks after the Opening Ceremony.
Initially 10 000 copies were sold and the number doubled after two weeks.

(i) Find the value of A. **1**

(ii) Calculate the value of k correct to 3 decimal places. **1**

(iii) At what rate was the number of copies increasing after four weeks? Answer correct to the nearest whole number. **2**

(b) Mr B—— borrows $\$P$ to fund his new Nissan supercar. The term of the loan is 10 years with an interest rate of 6% p.a., monthly reducible. He repays the loan in equal monthly installments of \$750.

(i) Show that at the end of n months, the amount owing is given by $A = P(1.005)^n - 150\,000(1.005)^n + 150\,000$. **3**

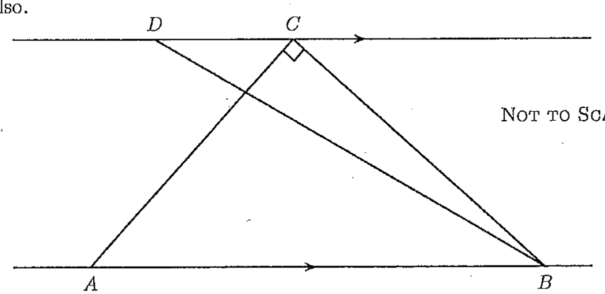
(ii) If at the end of 10 years the loan has been repaid, calculate the amount that he originally borrowed, correct to the nearest dollar. **2**

(c) A particle is moving along the x-axis. The distance of the particle, x metres from the origin O is given by the equation $x = 6t + e^{-4t}$ where t is time in seconds.

(i) Write down an expression for velocity of the particle. **1**

(ii) Explain why the particle will never come to rest. **2**

(d) A, B, and C are the vertices of an isosceles triangle where $AC = BC$ and right-angled at C. D is a point such that $DB = AB$ and $D\hat{B}A$ is acute. $DC \parallel AB$ also. **3**



Find, giving reasons, the size of $D\hat{B}C$.

End of Paper

Section I— 10 marks

Select the alternative A, B, C, or D that best answers the question.
Fill in the response oval on your multiple choice answer sheet.

Marks

1. $L + 2m - (L - 2n) - [2m + L - (2n - L)]$ simplifies to [1]
   (A) $4n + 2L$
   (B) $4n - 2L$
   (C) $4m + 4n - 2L$
   (D) $4m - 4n - 2L$

Solution: $L + 2m - L + 2n - (2m + L - 2n + L) = 2m + 2n - (2m - 2n + 2L)$
$= 4n - 2L$

2. At 10% p.a. simple interest, how long will it take for a sum of money to double? [1]
   (A) 7.3 years
   (B) 5 years
   (C) 7.27 years
   (D) 10 years

Solution: $I = P \times \frac{10}{100} \times n$
$= P$ (i.e. $2P = P + I$)
so $P = \dfrac{Pn}{10}$
$n = 10$

3. What is the value of k if the expression $4x^2 - 6x + k$ is a perfect square? [1]
   (A) $\dfrac{4}{9}$
   (B) $\dfrac{9}{4}$
   (C) 4
   (D) 9

Solution: $4\left(x^2 - \dfrac{6x}{4} + \dfrac{k}{4}\right) = 4\left(x^2 - \dfrac{3x}{2} + \dfrac{9}{16}\right)$
So $k = \dfrac{9}{4}$

4. $\dfrac{x^2+4x}{x^3-9x} \div \dfrac{x^2+2x-8}{x^2+x-6}$ simplifies to [1]
   (A) 1
   (B) $\dfrac{x}{x-3}$
   (C) $\dfrac{1}{x-3}$
   (D) $\dfrac{1}{x+3}$

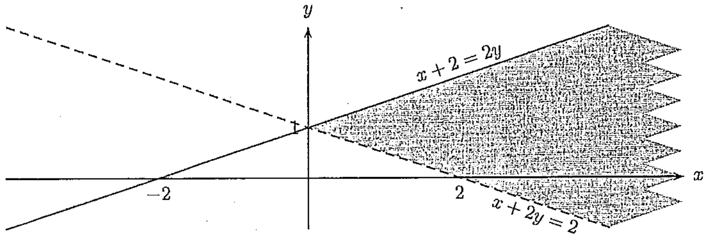
Solution: $\dfrac{x(x+4)}{x(x^2-9)} \times \dfrac{(x+3)(x-2)}{(x+4)(x-2)} = \dfrac{(x+3)}{(x+3)(x-3)}$
$= \dfrac{1}{x-3}$

5. The solution to the equation $2x^2 = 7x$ is $x =$ [1]
   (A) 0 or $-3\frac{1}{2}$
   (B) 0 or $3\frac{1}{2}$
   (C) $3\frac{1}{2}$ only
   (D) $3\frac{1}{2}$ or $-3\frac{1}{2}$

Solution: $x(2x - 7) = 0$
$\therefore x = 0$ or $3\frac{1}{2}$

6. If p and q are the roots of $15x^2 + 75x - 3 = 0$ then $p + q =$ [1]
   (A) 75
   (B) 5
   (C) $-\frac{1}{5}$
   (D) -5

Solution: $p + q = -\dfrac{75}{15}$
$= -5$

7. 

The shaded region in the diagram satisfies

(A) $x + 2 \geqslant 2y$ and $x + 2y > 2$

(B) $x + 2 \geqslant 2y$ and $x + 2y < 2$

(C) $x + 2 \leqslant 2y$ and $x + 2y > 2$

(D) $x + 2 \leqslant 2y$ and $x + 2y < 2$

[1]

Solution: Testing (2, 1) in $x + 2 \geqslant 2y$ gives true,
and testing (2, 1) in $x + 2y > 2$ gives true,
so the correct answer is (A).

8. $\log_3 15 + \log_3 18 - \log_3 10 =$

(A) 1

(B) 2

(C) 3

(D) 0

[1]

Solution:
$$\log_3\left(\frac{15 \times 18}{10}\right) = \log_3 27$$
$$= \log_3 3^3$$
$$= 3$$

9. Two cards are drawn in succession from a regular pack of 52 cards. What is the probability that both cards are diamonds or both cards are clubs? [1]

(A) $\frac{2}{17}$

(B) $\frac{3}{5}$

(C) $\frac{3}{17}$

(D) $\frac{27}{52}$

Solution: P(both diamonds) $= \frac{1}{4} \times \frac{12}{51} = \frac{1}{17}$,

$\therefore$ P(both diamonds or both clubs) $= \frac{1}{17} + \frac{1}{17} = \frac{2}{17}$

10. If the 5th term and 18th term of an arithmetic series are 12 and 64 respectively, find the common difference. [1]

(A) -5

(B) 4

(C) -4

(D) 5

Solution:
$$18 - 5 = 13$$
$$(64 - 12) \div 13 = 4$$

Section II— 90 marks

Marks

Question 11 (15 marks) (use a separate answer booklet)

(a) Find the first derivative of

(i) $y = (x^2 - 1)^3$, [2]

Solution: $3 \times 2x(x^2 - 1)^2 = 6x(x^2 - 1)^2$.

(ii) $y = \dfrac{2x}{x - 1}$, [2]

Solution: $\dfrac{(x-1).2 - 2x}{(x-1)^2} = \dfrac{-2}{(x-1)^2}$.

(iii) $f(x) = \ln(3 - x)$. [1]

Solution: $\dfrac{-1}{3 - x}$.

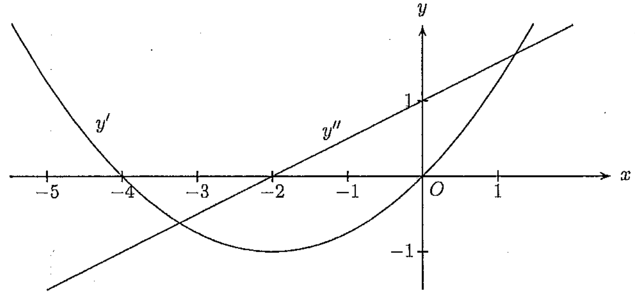
(b) Evaluate $\int_3^8 \sqrt{x+1}\,dx$. [3]

Solution:
$$\begin{aligned}\int_3^8 (x+1)^{1/2}.dx &= \left[\frac{2(x+1)^{3/2}}{3}\right]_3^8,\\ &= 18 - \frac{16}{3},\\ &= \frac{38}{3} \text{ or } 12\tfrac{2}{3}.\end{aligned}$$

(c) Find the equation of the normal to the curve $y = \tan x$ at the point where $x = \frac{\pi}{4}$ (answer in the general form of a line). [3]

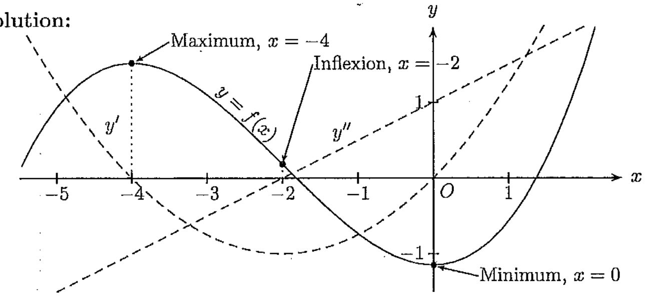
Solution: Point $(\frac{\pi}{4}, 1)$,
$y' = \sec^2 x$, $\therefore$ tangent slope $= 2$.
Hence normal is:
$$\begin{aligned} y - 1 &= -\tfrac{1}{2}(x - \tfrac{\pi}{4}),\\ 2y - 2 &= -x + \tfrac{\pi}{4},\\ x + 2y - 2 - \tfrac{\pi}{4} &= 0.\end{aligned}$$

(d) The graph shows y' and y'' for the function $y = f(x)$. [3]



Sketch a graph of $y = f(x)$, clearly showing the x values of any turning points and points of inflexion.

Solution:



(e) Find $\int 3\cos\left(\frac{x}{2}\right)\,dx$. [1]

Solution:
$$\begin{aligned}\int 3\cos\left(\frac{x}{2}\right)\,dx &= 2 \times 3\sin\left(\frac{x}{2}\right) + c,\\ &= 6\sin\left(\frac{x}{2}\right) + c,\end{aligned}$$

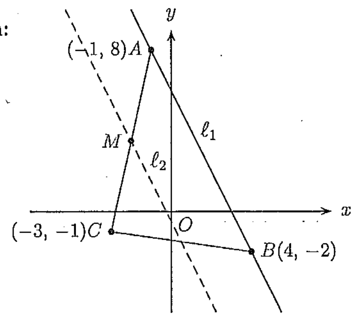
Marks

Question 12 (15 marks) (use a separate answer booklet)

(a) $A(-1, 8)$, $B(4, -2)$, and $C(-3, -1)$ are three points on the number plane. The line ℓ_1 passes through the points A and B.

(i) Draw a sketch showing A, B, C, and ℓ_1. [1]

Solution:



(ii) Find the exact distance AB. [1]

Solution: $AB = \sqrt{(4+1)^2 + (-2-8)^2}$,
$= \sqrt{25+100}$,
$= 5\sqrt{5}$.

(iii) Show that ℓ_1 has the equation $2x + y - 6 = 0$. [1]

Solution: Slope $= \dfrac{8+2}{-1-4}$,
$= -2$.
Equation: $y + 2 = -2(x-4)$,
$= -2x + 8$,
$\therefore 2x + y - 6 = 0$.

(iv) Find the perpendicular distance from the point C to the line $2x+y-6=0$. [2]

Solution: Perp. distance $= \dfrac{|2(-3)+(-1)-6|}{\sqrt{4+1}}$,
$= \dfrac{13}{\sqrt{5}}$ or $\dfrac{13\sqrt{5}}{5}$.

(v) Calculate the area of the triangle ABC. [2]

Solution: Area $= \frac{1}{2} \times 5\sqrt{5} \times \dfrac{14}{\sqrt{5}}$,
$= \dfrac{65}{2}$.

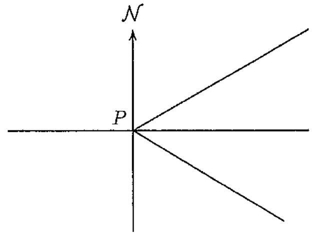
(vi) Find the co-ordinates of the midpoint, M, of AC. [1]

Solution: $M = \left(\dfrac{-1-3}{2}, \dfrac{8-1}{2}\right)$,
$= (-2, 7/2)$.

(vii) Find the equation of the line, ℓ_2, through M and parallel to AB (written in the general form of a line). [2]

Solution: ℓ_2 is $y - \frac{7}{2} = -2(x+2)$,
$2x + y + 1/2 = 0$.

(b) Two separate 'one man' canoes start off from a jetty, P, on a very large lake. The first canoeist paddles on a bearing of 040° T for 12 nautical miles to a buoy Q. At the same time the second canoeist paddles a distance of 8 nautical miles on a bearing of 100° T to another buoy R.



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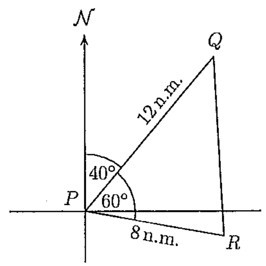
(i) Copy the sketch above and add all the relevant information. [1]

Solution:



(ii) Calculate the distance (in nautical miles) between the canoeists correct to one decimal place. [2]

> **Solution:** $p^2 = 8^2 + 12^2 - 2.8.12.\cos 60°$,
> $= 112$.
> $\therefore$ Distance between is 10.6 n.m.

(iii) If the two canoeists conduct a quick search of $\triangle PQR$ for any other canoeists, calculate the total area searched, giving your answer in square kilometres correct to the nearest 10 square kilometres. (Note: 1 nautical mile = 1852 metres.) [2]

> **Solution:** Area $= \frac{1}{2} \times 8 \times 12 \times \sin 60° \times (1.852)^2$,
> $\approx 140\,\text{km}^2$ (nearest $10\,\text{km}^2$).

Marks

Question 13 (15 marks) (use a separate answer booklet)

(a) (i) For the curve $y = 3\sin 4x$ in the domain $0 \leqslant x \leqslant \pi$, state the

(α) period, [1]

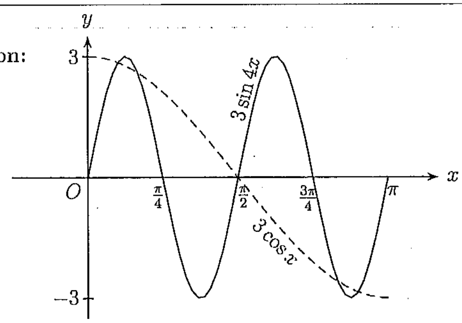
> **Solution:** $\frac{\pi}{2}$.

(β) amplitude. [1]

> **Solution:** 3.

(ii) Sketch the curve $y = 3\sin 4x$, $0 \leqslant x \leqslant \pi$, clearly showing where the curve cuts the x-axis. [2]

Solution:



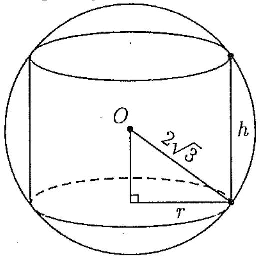
(iii) Hence or otherwise, find the NUMBER of solutions to $\sin 4x = \cos x$ where $0 \leqslant x \leqslant \pi$. [2]

> **Solution:** Multiply throughout by 3 gives $3\sin 4x = 3\cos x$,
> then graph $y = 3\cos x$ on the diagram above.
> It is clear that there are 5 solutions.

(b) Evaluate $\displaystyle\int_0^1 2xe^{(3x^2-5)}\,dx$, giving your answer to 3 significant figures. [3]

> **Solution:** Note that $\frac{d}{dx}\left(e^{(3x^2-5)}\right) = 6xe^{(3x^2-5)}$.
> So $\frac{1}{3}\displaystyle\int_0^1 6xe^{(3x^2-5)}\,dx = \frac{1}{3}\left[e^{(3x^2-5)}\right]_0^1$,
> $= \frac{1}{3}\left(e^{-2} - e^{-5}\right)$,
> ≈ 0.0429 (3 sig. fig.)

(c) A right circular cylinder of radius r cm and height h cm has to be designed to fit inside a sphere of $2\sqrt{3}$ cm radius so that both the bottom and the top touch the sphere (centre O) completely on the circular rim.



(i) Using the diagram as a guide, show that $r^2 = 12 - \dfrac{h^2}{4}$. [1]

Solution:
$$\left(\frac{h}{2}\right)^2 + r^2 = (2\sqrt{3})^2 \text{ (by Pythagoras's Thm.)},$$
$$r^2 = 4 \times 3 - \frac{h^2}{4},$$
$$= 12 - \frac{h^2}{4}.$$

(ii) If the volume of the cylinder is V, show that $V = 12\pi h - \frac{1}{4}\pi h^3$. [1]

Solution:
$$V = \pi r^2 h,$$
$$= \pi h\left(12 - \frac{h^2}{4}\right),$$
$$= 12\pi h - \frac{\pi h^3}{4}.$$

(iii) Hence find the dimensions of the cylinder to give maximum volume. [2]

Solution:
$$\frac{dV}{dh} = 12\pi - \frac{3\pi h^2}{4},$$
$$= 0 \text{ when } 12\pi = \frac{3\pi h^2}{4},$$
$$48\pi = 3\pi h^2,$$
$$h^2 = 16,$$
$$h = 4\,\text{cm}.$$
$$\frac{d^2V}{dh^2} = -\frac{6\pi h}{4},$$
$$= -\frac{3\pi h}{2},$$
$$= -6\pi \text{ when } h = 4,$$
$$< 0 \Longrightarrow \text{maximum}.$$
$$\text{So } r^2 = 12 - \frac{16}{4},$$
$$= 8,$$
$$r = 2\sqrt{2}\,\text{cm}.$$

(d) Evaluate $\displaystyle\int_0^5 \frac{x}{5+3x^2}\,dx$, leaving your answer in exact form. [2]

Solution:
$$\frac{1}{6}\int_0^5 \frac{6x}{5+3x^2}\,dx = \frac{1}{6}\Big[\ln(5+3x^2)\Big]_0^5,$$
$$= \frac{1}{6}\ln\frac{80}{5},$$
$$= \frac{1}{3}\ln 4.$$

Marks

Question 14 (15 marks) (use a separate answer booklet)

(a) Given that $x^2 - 6x - 7 = 8y$, find:

(i) the co-ordinates of the vertex, [1]

> **Solution:** $x^2 - 6x + 3^2 = 8y + 7 + 9,$
> $(x-3)^2 = 4 \times 2(y+2).$
> So the vertex is $(3, -2)$.

(ii) the co-ordinates of the focus, [1]

> **Solution:** From above, the focus is $(3, 0)$.

(iii) the equation of the directrix. [1]

> **Solution:** Also from above, the directrix is $y = -4$.

(b) A road grader removes V m^3 of soil in t minutes, where $V = 25t - \frac{t^2}{50}$. [2]
Find the rate at which the soil is being removed after five minutes.

> **Solution:** $\frac{dV}{dt} = 25 - \frac{2t}{50}.$
> When $t = 5$, $\frac{dV}{dt} = 25 - \frac{5}{25},$
> $= 24\frac{4}{5}.$
> $\therefore$ The rate of removal is $24\frac{4}{5}$ m^3/min.

(c) A driver in a car is at a point A, from which branches out two roads. If he takes the road on the LEFT and journeys some distance, this road leads to a point B from which branches off three roads, one of which leads to his destination C. However if he takes the road on the RIGHT, and journeys along a certain distance, this road leads to a point D, from which branches off four roads, one of which leads to his destination C. [3]
Assuming he has no Sat. Navigation or prior knowledge of any of theses facts, except that he wants to travel to destination C, find the probability that he DOES NOT reach C on his first try.

> **Solution:** $\text{P}(C \text{ via } B) = \frac{1}{2} \times \frac{1}{3},$
> $= \frac{1}{6}.$
> $\text{P}(C \text{ via } D) = \frac{1}{2} \times \frac{1}{4},$
> $= \frac{1}{8}.$
> $\text{P}(\widetilde{C}) = 1 - \left(\frac{1}{6} + \frac{1}{8}\right),$
> $= \frac{17}{24}.$

(d) Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx + \int_0^{\frac{\pi}{2}} \cos^2 x\,dx.$ [2]

> **Solution:** $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx + \int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \int_0^{\frac{\pi}{2}} 1\,dx,$
> $= x\Big]_0^{\frac{\pi}{2}},$
> $= \frac{\pi}{2}.$

(e) Let $A(4, 0)$ and $B(1, 0)$ be two fixed points and let P be the variable point (x, y) .

(i) Write down expressions for the distances PA and PB in terms of x and y. [2]

> **Solution:** $PA = \sqrt{(x-4)^2 + y^2}, \qquad PB = \sqrt{(x-1)^2 + y^2}$

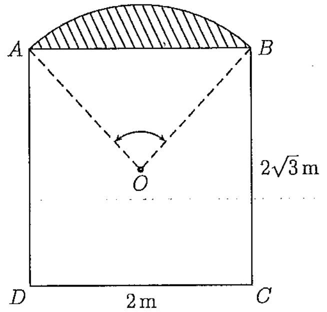
(ii) Find the locus of P whose distance from A is twice its distance from B. [3]

> **Solution:** $PA = 2PB,$
> $PA^2 = 4PB^2,$
> $(x-4)^2 + y^2 = 4\{(x-1)^2 + y^2\},$
> $x^2 - 8x + 16 + y^2 = 4\{x^2 - 2x + 1 + y^2\},$
> $3x^2 + 3y^2 = 12,$
> $x^2 + y^2 = 2^2.$
> So the locus is a circle with centre at the origin and radius 2.

Marks

Question 15 (15 marks) (use a separate answer booklet)

(a) The diagram below shows an ancient window which consists of a rectangle $ABCD$ with height $2\sqrt{3}$ m and width 2 m surmounted by a minor segment of a circle which is stained glass. The centre of the circle is at O, the point of intersection of the diagonals of the rectangle.



(i) Explain why $A\widehat{O}B = 60°$. [2]

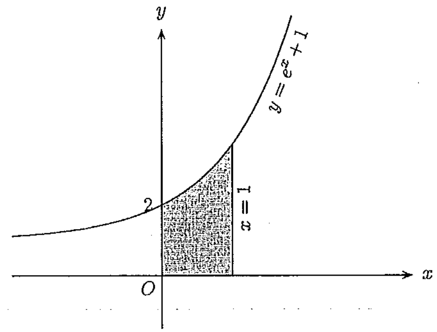
> **Solution:** $DB = \sqrt{4+12}$,
> $= 4$ m.
> O bisects DB, $\therefore OB = 2$ m.
> Similarly $OB = 2$ m and, $AB = 2$ m,
> $\triangle OAB$ is equilateral, so $A\widehat{O}B = 60°$.

(ii) Find the area of the minor segment correct to 3 decimal places. [2]

> **Solution:** Minor segment $=$ sector $AOB - \triangle AOB$,
> $= \frac{1}{6} \times \pi \times 2^2 - \frac{1}{2} \times 2^2 \times \sin 60°$,
> $\approx 0.362\,\text{m}^2$ (3 d.p.)

(b) (i) Sketch the region beneath the curve $y = e^x + 1$ which is above the x-axis and between the lines $x = 0$ and $x = 1$. [1]

> **Solution:**



(ii) The region in (b)(i) is now rotated about the x-axis. Find the volume of the resulting solid of revolution. Leave your answer in exact form. [3]

> **Solution:**
> $$\begin{aligned} \text{Vol.} &= \pi \int_0^1 y^2\,dx, \\ &= \pi \int_0^1 \left(e^{2x} + 2e^x + 1\right) dx, \\ &= \pi \left[\frac{e^{2x}}{2} + 2e^x + x\right]_0^1, \\ &= \pi \left\{\frac{e^2}{2} + 2e + 1 - \left(\frac{1}{2} + 2 + 0\right)\right\}, \\ &= \pi \left(\frac{e^2}{2} + 2e - \frac{3}{2}\right). \end{aligned}$$

(c) For the curve $y = xe^{-x}$,

(i) *Prove* that $\dfrac{dy}{dx} = -e^{-x}(x-1)$. [1]

> **Solution:**
> $$\begin{aligned} \frac{dy}{dx} &= 1 \times e^{-x} + x \times (-1) \times e^{-x} \text{ (using the product rule)}, \\ &= e^{-x}(1-x), \\ &= -e^{-x}(x-1). \end{aligned}$$

(ii) Find any stationary points and determine their nature. [2]

> **Solution:** $\dfrac{dy}{dx} = 0$ when $x = 1$.
>
> $\therefore$ Maximum at $(1, {}^1\!/_e)$.

	0	1	2
$\frac{dy}{dx}$	1	0	−0.135
	↗	⟶	↘

(iii) *Prove* that $\dfrac{d^2y}{dx^2} = e^{-x}(x-2)$. [1]

Solution:
$$\begin{aligned}\frac{d^2y}{dx^2} &= -(-1)(e^{-x})(x-1) + 1 \times (-e^{-x}) \text{ (product rule)},\\ &= e^{-x}(x-1) - e^{-x},\\ &= e^{-x}(x-2).\end{aligned}$$

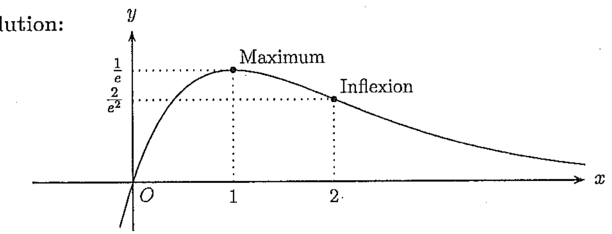
(iv) Show that there is a point of inflexion on this curve and find the co-ordinates of this point. [2]

Solution: $\dfrac{d^2y}{dx^2} = 0$ when $x = 2$.
Change of concavity.
$\therefore$ Inflexion at $\left(2, \dfrac{2}{e^2}\right)$.

	1	2	3
$\frac{d^2y}{dx^2}$	−0.37	0	0.05
	$\downarrow$		$\uparrow$

(v) Sketch the curve, showing the co-ordinates of the point of inflexion and any stationary points. [1]

Solution:



Marks

Question 16 (15 marks) (use a separate answer booklet)

(a) The number of DVD copies sold at a store of *The London Olympics 2012 Opening Ceremony* has increased exponentially in accordance with the formula $N = Ae^{kt}$ where t is the time in weeks after the Opening Ceremony.
Initially 10 000 copies were sold and the number doubled after two weeks.

(i) Find the value of A. [1]

Solution:
$$\begin{aligned}N &= Ae^{kt},\\ 10\,000 &= Ae^{0},\\ \therefore A &= 10\,000.\end{aligned}$$

(ii) Calculate the value of k correct to 3 decimal places. [1]

Solution:
$$\begin{aligned}20\,000 &= 10\,000e^{2k},\\ e^{2k} &= 2,\\ k &= \ln\sqrt{2},\\ &\approx 0.347.\end{aligned}$$

(iii) At what rate was the number of copies increasing after four weeks? Answer correct to the nearest whole number. [2]

Solution: $\dfrac{dN}{dt} = kAe^{kt}$.
$\therefore$ After 4 weeks, $\dfrac{dN}{dt} = 0.347 \times 10\,000e^{4\times 0.347}$,
$\approx 13\,863$ (nearest integer).
So sales were increasing at 13 863 copies/week.

(b) Mr B—— borrows \$$P$ to fund his new Nissan supercar. The term of the loan is 10 years with an interest rate of 6% p.a., monthly reducible. He repays the loan in equal monthly installments of \$750.

(i) Show that at the end of n months, the amount owing is given by $A = P(1.005)^n - 150\,000(1.005)^n + 150\,000$. [3]

Solution: 6% a year is equivalent to a monthly rate of 0.5%.
$$\begin{aligned}\text{Owe after 1 mo.} &= P(1.005) - 750,\\ \text{owe after 2 mo.} &= (P(1.005) - 750)(1.005) - 750,\\ \text{owe after 3 mo.} &= ((P(1.005) - 750)(1.005) - 750)(1.005) - 750,\\ &= P(1.005)^3 - 750(1 + 1.005 + 1.005^2),\\ \text{owe after } n \text{ mo.} &= P(1.005)^n - 750(1 + 1.005 + \cdots + 1.005^{n-1}),\\ &= P(1.005)^n - \frac{750(1.005^n - 1)}{1.005 - 1},\\ &= P(1.005)^n - 150\,000(1.005^n - 1),\\ &= P(1.005)^n - 150\,000(1.005)^n + 150\,000.\end{aligned}$$

(ii) If at the end of 10 years the loan has been repaid, calculate the amount that he originally borrowed, correct to the nearest dollar. [2]

Solution:
$$0 = P(1.005)^{120} - 150\,000(1.005)^{120} + 150\,000,$$
$$P = \frac{150\,000(1.005^{120} - 1)}{1.005^{120}},$$
$$\approx 67\,555.$$
His car cost \$67 555.

(c) A particle is moving along the x-axis. The distance of the particle, x metres from the origin O is given by the equation $x = 6t + e^{-4t}$ where t is time in seconds.

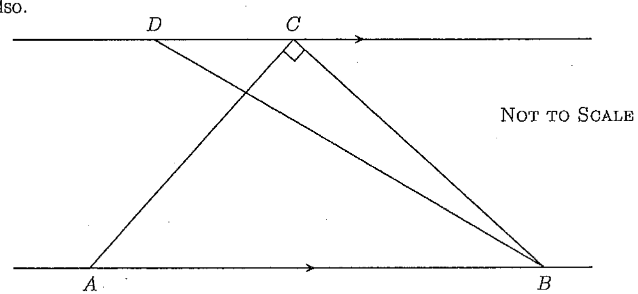
(i) Write down an expression for velocity of the particle. [1]

Solution: Let velocity be v.
$$x = 6t + e^{-4t},$$
$$v = \frac{dx}{dt},$$
$$= 6 - 4e^{-4t}.$$

(ii) Explain why the particle will never come to rest. [2]

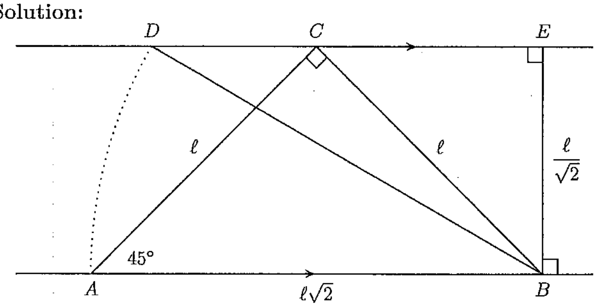
Solution: When $t = 0$, $v = 2$,
as $t \to \infty$, $v \to 6$.
Acceleration, $\frac{dv}{dt} = 16e^{-4t}$ which is always positive.
So velocity is always positive and the particle is always accelerating away from the origin and can never come to rest.

(d) A, B, and C are the vertices of an isosceles triangle where $AC = BC$ and right-angled at C. D is a point such that $DB = AB$ and $D\widehat{B}A$ is acute. $DC \parallel AB$ also. [3]



Find, giving reasons, the size of $D\widehat{B}C$.

Solution:



Let $AC = \ell = BC$.
$$\ell^2 + \ell^2 = AB^2 \text{ (Pythagoras)},$$
$$\therefore AB = \sqrt{2\ell^2} = \ell\sqrt{2},$$
$$C\widehat{A}B = C\widehat{B}A = 45^\circ \text{ (equal base } \angle\text{s of isosc. } \triangle).$$
Draw $BE \perp AB$ and DE ($AB \parallel DE$),
$$C\widehat{B}E = 45^\circ \text{ (complement of } C\widehat{B}A),$$
$$E\widehat{C}B = 45^\circ \text{ } (\angle \text{ sum } \triangle CEB),$$
$\triangle CEB$ is isosceles (equal base angles),
$$CE = EB \text{ (sides opposite equal } \angle\text{s)},$$
$$2EB^2 = \ell^2,$$
$$EB = \frac{\ell}{\sqrt{2}},$$
$$\cos E\widehat{B}D = \frac{\ell}{\sqrt{2}} \times \frac{1}{\ell\sqrt{2}},$$
$$= \frac{1}{2},$$
$$E\widehat{B}D = 60^\circ.$$
$$\therefore C\widehat{B}D = E\widehat{B}D - C\widehat{B}E,$$
$$= 60^\circ - 45^\circ,$$
$$= 15^\circ.$$

End of Paper