2012 Assessment Task 2

MATHEMATICS

Extension 1

Year 12

Time allowed - 60 minutes (plus 5 minutes reading time)

Topics: Logarithmic and Exponential Functions, Trigonometric Functions I and II

Instructions

Name and Class ____

- Attempt all questions.
- Questions are of equal value.
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- Write on one side of the paper only.
- Questions do not necessarily appear in order of difficulty.
- Diagrams are not to scale
- A table of standard integrals is attached

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

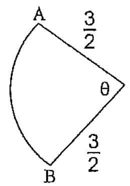
NOTE : $\ln x = \log_e x, \quad x > 0$

Question One (12 marks)

a) Evaluate $\log_2 8$ [1]

b) Find the equation of the tangent to the curve $y = e^{2x}$ at the point where $x = 1$ [3]

c) The area of the sector below is $\frac{3\pi}{8}cm^2$ and its radius is $\frac{3}{2}cm$



i) Find the size of the angle θ in radians [2]

ii) Hence find the exact length of arc AB [1]

d) Find $\int \frac{x-1}{x^2-2x+3}dx$ [2]

e) For what value of x is $\log_3(x+1) - \log_3 x = 2$ [3]

Question Two (12 Marks)

a) Find the derivatives of:

i) $y = e^{2\sin x}$ [1]

ii) $y = \sec x$ [1]

b) If $\tan\frac{\theta}{2} = 2$, find the exact value of $\sin\theta$ [2]

c) i) Sketch $y = 3\sin\frac{x}{2}$ for $-\pi \le x \le \pi$ [2]

ii) Hence find the area bounded by the curve, the X-axis and $x = \pm\frac{\pi}{2}$ (3)

d) Solve $2\sin x\cos x - \cos x = 0$ for $0 \le x \le 2\pi$ [3]

Question Three (12 marks)

a) i) Find $\frac{d}{dx}(\log_e x)^2$ [1]

ii) Hence evaluate $\int_1^2 \frac{\log_e x}{x}dx$ [2]

b) Without the use of a calculator show that $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{4}$ [3]

c) i) Sketch the curve $y=\log_e(x+2)$ showing all key features [2]

ii) The area bounded by $y=\log_e(x+2)$ and the axes is rotated about the Y- axis. Find the exact volume of the solid generated [4]

Question Four (12 Marks)

a) Find $\int e^{-3x}dx$ [1]

b) Find $\int \sin^2 x dx$ [2]

c) Find the sum of $\log_a \frac{1}{x}+\log_a \frac{1}{x^2}+\log_a \frac{1}{x^3}+\dots\dots\dots\dots+\log_a \frac{1}{x^6}$ for $x>1$ [2]

d) Evaluate $\lim_{x\to 0}\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{x}$ [2]

e) i) Write $2\sin x+\cos x$ in the form $r\sin(x+\alpha)$ [2]

ii) Hence solve $2\sin x+\cos x=1$ for $0^\circ \le x \le 360^\circ$ [3]

Question Five (12 Marks)

a) Find the size of the acute angle between the lines with gradients $m_1 = 3$ and $m_2 = \frac{7}{2}$ [2]

b) i) Show that $\frac{\sin 2\theta}{1-\cos 2\theta} = \cot\theta$ [2]

ii) Hence find the exact value of $\cot 15^\circ$ [1]

c) Find $\int 2x^2 e^{x^3}\,dx$ [2]

d) Show that $\frac{d}{dx}\log_e\left(\frac{1+\sin x}{\cos x}\right) = \sec x$ [2]

e) Find $\int \sqrt{1-\sin 2x}\,dx$ given that $0 < x < \frac{\pi}{4}$ (justify your answer) [3]

Question One (12 Marks)

a) $\log_2 8 = x$ ie $2^x = 8$

$x = 3$ (1)

b) $y = e^{2x}$

$\frac{dy}{dx} = 2e^{2x}$

when $x = 1$

$m = 2e^2$, $y = e^2$

$y - y_1 = m(x - x_1)$

$y - e^2 = 2e^2(x-1)$

$y - e^2 = 2xe^2 - 2e^2$

$y = 2xe^2 - e^2$ (3)

c) i) $A = \frac{1}{2} r^2 \theta$ ii) $l = r\theta$

ie $\frac{3\pi}{8} = \frac{1}{2}\left(\frac{3}{2}\right)^2 \theta$ $= \frac{3}{2} \times \frac{\pi}{3}$ i) (2)

$= \frac{\pi}{2}$ cm

$\frac{3\pi}{8} = \frac{9\theta}{8}$ ii) (1)

$\theta = \frac{3\pi}{8} \times \frac{8}{9}$

$= \frac{\pi}{3}$

d) $\int \frac{x-1}{x^2 - 2x + 3}\,dx = \frac{1}{2}\int \frac{2x-1}{x^2-2x+3}\,dx$

$= \frac{1}{2}\log_e(x^2 - 2x + 3) + c$

or $\log_e \sqrt{x^2 - 2x + 3}$ (2)

e) $\log_3(x+1) - \log_3 x = 2$ (3)

$\log_3 \frac{x+1}{x} = 2$

$3^2 = \frac{x+1}{x}$

$9 = \frac{x+1}{x}$

$9x = x+1$

$8x = 1$ $x = \frac{1}{8}$

Question Two (12 marks)

a) $y = e^{2\sin x}$

i) $\frac{dy}{dx} = (2\cos x)(e^{2\sin x})$ (1)

ii) $y = \sec x$

$= (\cos x)^{-1}$

$\frac{dy}{dx} = -(\cos x)^{-2}(-\sin x)$

$= \frac{\sin x}{\cos^2 x}$

$= \tan x \sec x$ (1)

b) $\tan\frac{\theta}{2} = 2$ ie $t = 2$

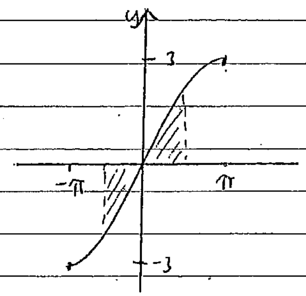
$\sin\theta = \frac{2t}{1+t^2}$

$= \frac{4}{5}$ (2)

c) i) $y = 3\sin\frac{x}{2}$

amp $= 3$ period $= \frac{2\pi}{\frac{1}{2}}$

$= 4\pi$



(2)

ii) $A = 2\int_0^{\frac{\pi}{2}} 3\sin\frac{x}{2}\,dx$

$= \left[6x - 2\cos\frac{x}{2}\right]_0^{\frac{\pi}{2}}$

$= -12\left[\cos\frac{x}{2}\right]_0^{\frac{\pi}{2}}$

$= -12\left[\cos\frac{\pi}{4} - \cos 0\right]$

$= -12\left[\frac{1}{\sqrt{2}} - 1\right]$

$= -6\sqrt{2} + 12$ units^2

$(12 - 6\sqrt{2})$ units^2

Q2 continued

d) $2\sin x\cos x - \cos x = 0$

$\cos x\,(2\sin x - 1) = 0$

$\cos x = 0$ or $\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ (3)

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

Question 3 (12 marks)

a) i) $\frac{d}{dx}(\log_e x)^2 = 2\log_e x \times \frac{1}{x}$

$= \frac{2\log_e x}{x}$ (1)

ii) $\int_1^2 \frac{\log_e x}{x}\,dx$

Now $\frac{d}{dx}(\log_e x)^2 = \frac{2\log_e x}{x}$

$\therefore \int \frac{2\log_e x}{x} = (\log_e x)^2$

$\therefore \int \frac{\log_e x}{x} = \frac{1}{2}(\log_e x)^2 + c$ (2)

$\int_1^2 \frac{\log_e x}{x}\,dx = \frac{1}{2}\left[(\log_e x)^2\right]_1^2$

$= \frac{1}{2}\left[(\log_e 2)^2 - (\log_e 1)^2\right]$

$= \frac{1}{2}(\log_e 2)^2$

b) let $x = \tan^{-1}\frac{1}{2}$, $y = \tan^{-1}\frac{1}{3}$

ie $\tan x = \frac{1}{2}$, ie $\tan y = \frac{1}{3}$

prove $x + y = \frac{\pi}{4}$ (3)

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

$= \left(\frac{1}{2} + \frac{1}{3}\right) \div \left(1 - \frac{1}{2}\times\frac{1}{3}\right)$

$= 1$

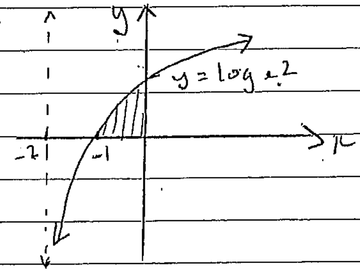
ie

$\tan(x+y) = 1$

$x + y = \tan^{-1} 1$

$= \frac{\pi}{4}$

c) i) $y = \log_e(x+2)$ (2)



ii) $y = \log_e(x+2) \Rightarrow x = e^y - 2$

$V = \pi\int_0^{\log_e 2} (e^y - 2)^2\,dy$ (4)

$= \pi\int_0^{\log_e 2} (e^{2y} - 4e^y + 4)\,dy$

$= \pi\left[\frac{1}{2}e^{2y} - 4e^y + 4y\right]_0^{\log_e 2}$

$= \pi\left[\frac{1}{2}e^{2\log_e 2} - 4e^{\log_e 2} + 4\log_e 2 - \frac{1}{2} + 4 - 0\right]$

$= \pi\left[\frac{1}{2}\times 4 - 4\times 2 + 4\log_e 2 + 3\frac{1}{2}\right]$

$V = \pi\left[4\log_e 2 - \frac{5}{2}\right]$ units^3

Question 4 (12 marks)

a) $\int e^{-3x}\,dx = -\frac{1}{3}e^{-3x} + c$ (1)

b) $\int \sin^2 x\,dx = \frac{1}{2}\int(1 - \cos 2x)\,dx$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$ (2)

c) $\log_a\frac{1}{x} + \log_a\frac{1}{x^2} + \log_a\frac{1}{x^3} + \dots + \log_a\frac{1}{x^6}$

$= \log_a x^{-1} + \log_a x^{-2} + \log_a x^{-3} + \dots + \log_a x^{-6}$

$= -\log_a x - 2\log_a x - 3\log_a x + \dots - 6\log_a x$

$= -21\log_a x$ (2)

d) $\lim_{x\to 0} \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{x}$

$= \lim_{x\to 0} \frac{\sin x}{x}$

$= 1$ (1)

e) i) $2\sin x + \cos x = r\sin(x+\alpha)$

$r = \sqrt{4+1}$

$= \sqrt{5}$

$2\sin x + \cos x = \sqrt{5}\sin(x+\alpha)$

ii) $\frac{2}{\sqrt{5}}\sin x + \frac{1}{\sqrt{5}}\cos x = \sin x\cos\alpha + \cos x\sin\alpha$

$\cos\alpha = \frac{2}{\sqrt{5}}$

$\sin\alpha = \frac{1}{\sqrt{5}}$

$\tan\alpha = \frac{1}{\sqrt{5}} \times \frac{\sqrt{5}}{2}$

$= \frac{1}{2}$

$\alpha = 26°34'$

$2\sin x + \cos x = 1$

ie $\sqrt{5}\sin(x+26°34') = 1$

$\sin(x+26°34') = \frac{1}{\sqrt{5}}$

$(x+26°34') = \frac{1}{\sqrt{5}}$ $(26°34' \leq x+26°26' \leq 386°34')$ (2)

$x+26°34 = 26°34', 153°26', 386°34'$

$x = 0, 126°52, 360°$ (3)

Question Five (12 marks)

a) $\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$

$= \left|\left(3 - \frac{7}{2}\right) \div \left(1 + \frac{21}{2}\right)\right|$

$= \frac{1}{23}$

$\theta = 2°$ (2)

b) i) $\frac{\sin 2\theta}{1-\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1-(1-2\sin^2\theta)}$

$= \frac{2\sin\theta\cos\theta}{2\sin^2\theta}$

$= \frac{\cos\theta}{\sin\theta}$

$= \cot\theta$ (2)

ii) $\cot 15° = \frac{\sin(2\times 15°)}{1-\cos(2\times 15°)}$

$= \frac{\sin 30°}{1-\cos 30°}$

$= \frac{\frac{1}{2}}{1-\frac{\sqrt{3}}{2}}$

$= \frac{1}{2-\sqrt{3}}$ (1)

c) $\int 2x^2 e^{x^3}\,dx = \frac{2}{3}\int 3x^2 e^{x^3}\,dx$

$= \frac{2}{3}e^{x^3} + C$ (2)

d) P.T.O

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d) $\frac{d}{dx} \log_e\left(\frac{1+\sin x}{\cos x}\right)$

$= \frac{d}{dx}\left(\log_e(1+\sin x) - (\log_e \cos x)\right)$

$= \frac{\cos x}{1+\sin x} + \frac{+\sin x}{\cos x}$

$= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos x(1+\sin x)}$

$= \frac{\cancel{(1+\sin x)}}{\cos x\cancel{(1+\sin x)}}$

$= \sec x$

e) $\sqrt{1-2\sin x} = \sqrt{\cos^2 x + \sin^2 x - 2\sin x\cos x}$

$= \sqrt{\cos^2 x - 2\sin x\cos x + \sin^2 x}$

$= \sqrt{(\cos x - \sin x)^2}$

$= \pm(\cos x - \sin x)$

Now for $0 < x < \frac{\pi}{4}$, $\cos x > \sin x$

$\therefore \int(\cos x - \sin x)\,dx$

$= \sin x + \cos x + C$