# Sydney Girls' High School

## 2012
## MATHEMATICS

## YEAR 12
## ASSESSMENT TASK 2

**Time Allowed: 90 minutes + 5minutes reading time**

**TOPICS:** Series & Sequences, The quadratic Polynomial,
The second Derivative & Applications of the Calculus.

**Directions to Candidates**

- There are five (5) questions.
- Attempt ALL questions.
- Questions are of equal value.
- Start each question on a new page.
- Write on one side of the paper only.
- Show all necessary working.
- Marks may be deducted for careless or badly arranged work.
- Diagrams are NOT drawn to scale.
- Board-approved calculators may be used.

**Total: 85 marks**

QUESTION 1 (17 marks) — Marks

(a) Given $y = x^2 + x - 6$

(i) Find the axis of symmetry of the parabola — 2

(ii) Find the vertex of the parabola — 1

(iii) For what values of $x$ is the parabola positive. — 2

(b) The sum of the first $n$ terms of a certain series is given by: $S_n = \dfrac{n(3n+1)}{2}$

(i) Calculate $S_1$ and $S_2$ — 2

(ii) Find the first three terms of this series — 3

(iii) Find an expression for the $n^{th}$ term — 2

(c) Find the equation of the tangent to the curve $y = x^3 - 3x^2$ where $\dfrac{d^2y}{dx^2} = 0$ — 5

QUESTION 2 (17 marks)

(a) The roots of $4x^2 - 12x - 1 = 0$ are $\alpha$ and $\beta$.

Find the value of : (i) $\alpha + \beta$ — 1

(ii) $\alpha\beta$ — 1

(iii) $5\alpha^2\beta + 5\alpha\beta^2$ — 2

(iv) $\dfrac{2}{\alpha^2} + \dfrac{2}{\beta^2}$ — 3

(b) For what values of $m$ will the equation $2x^2 + 4mx - (m-1) = 0$

(i) no real roots — 3

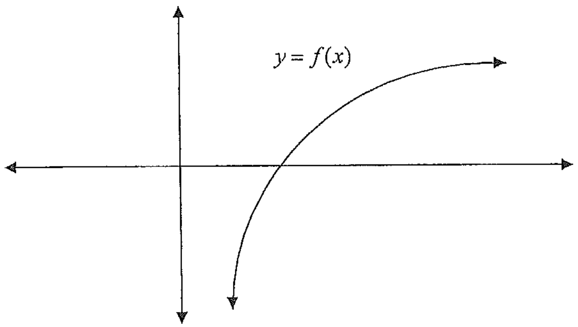
(ii) one root is the reciprocal of the other — 2

## QUESTION 2(continued) (17 marks)

Marks

(c) Calculate the values of $a, b$ and $c$ if $2x^2+3x-9 \equiv ax(x-1)+b(x-1)+c$ for all values of $x$. **3**

(d) Given the graph of $y=f(x)$ below state the sign of $f'(x)$ and $f''(x)$ **2**




## QUESTION 3 (17 marks)

(a) Sadia invested \$20 000 with a finance company for 8 years.
The investment offered monthly compound interest at an annual rate of 16%.

(i) What was the value of her investment at the end of 8 years? **2**

(ii) How much interest did Sadia earn in that time? **1**

(iii) Calculate the equivalent rate of simple interest. **2**

(b) Given the curve $y=7+4x^3-3x^4$

(i) Find the co-ordinates of the stationary points. **3**

(ii) Determine the nature of the stationary points. **4**

(iii) Find all values of $x$ for which $\frac{d^2y}{dx^2}=0$. **2**

(iv) Sketch the curve for the domain $-1 \le x \le 2$. **2**

(v) For what values of $x$ in this domain is the curve concave up? **1**

## QUESTION 4 (17 marks)

Marks

(a) Calculate the limiting sum of the geometric series in which $T_3=-24$ and $T_6=3$. **3**

(b) For what values of $m$ is the line $y=m(x-1)$ a tangent to the parabola $y=2x^2$. **3**

(c) Find the second derivative of $y=\sqrt{2x+1}$ **3**

(d) Jane borrows \$15 000 over three years at a 1% per month reducible interest rate.

Interest is compounded monthly. Jane makes a monthly repayment of \$M.

(i) Write down the balance owing at the end of the first month after interest has been added and then the first repayment deducted. **1**

(ii) Hence, or otherwise, write down a formula for the balance owing after $n$ months. **2**

(iii) Find the value of \$M, the monthly repayment. **2**

(iv) Find the amount of interest paid. **1**

(v) Hence, find the equivalent simple interest rate. **2**

## QUESTION 5 (17 marks)

(a) For the sequence $P_n=2^{n-1}+5$

(i) Find the first and seventh term **2**

(ii) Which term is equal to 261? **3**

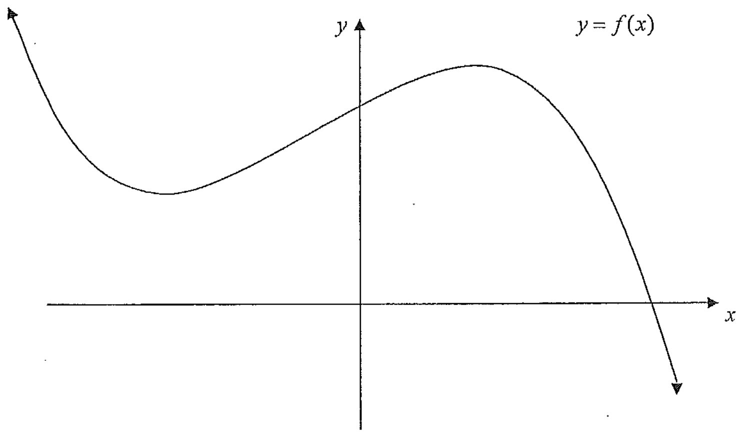
(b) The roots of the equation $x^2+px+4=0$ are $\alpha$ and $\alpha+3$.

Find the values of $\alpha$ and $p$. **4**

(c) Below is a graph of the curve $y = f(x)$

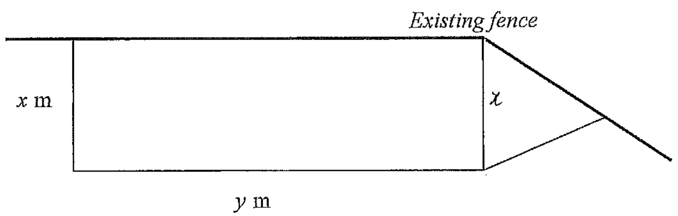
(i) Copy this graph onto your answer paper.

(iii) On the same set of axes sketch the graph $y = f'(x)$ 1




(d) Joanne has 1500 metres of fencing and wishes to enclose two fields, a rectangular field and a field in the shape of an equilateral triangle. Both fields have one side formed by an existing fence and one fence in common as shown in the diagram.

Let the rectangular field be $y$ metres and its width be $x$ metres.




(i) Show $y = 1500 - 3x$ 1

(ii) Show $A = 1500x - x^2(3 - \frac{\sqrt{3}}{4})$ where $A$ is the area enclosed by the two fields. 2

(iii) Find the dimensions of the rectangle which maximize the total area enclosed (give answer correct to 1 decimal place). 4

THE END

Mathematics – 2012 – Yr 12 Assessment Task 2

## Question 1

a) $y = x^2 + x - 6$

i) $x = -\frac{b}{2a}$

$x = \frac{-1}{2}$

ii) Vertex at $x = -\frac{1}{2}$

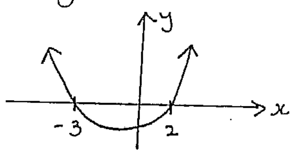
$y = (-\frac{1}{2})^2 - \frac{1}{2} - 6$

$y = -6\frac{1}{4}$

$\therefore$ Vertex $(-\frac{1}{2}, -6\frac{1}{4})$

iii) $y = x^2 + x - 6$

$y = (x+3)(x-2)$




$(x+3)(x-2) > 0$

$x > 2$ and $x < -3$

b) $S_n = n\frac{(3n+1)}{2}$

i) $S_1 = 1\frac{(3+1)}{2}$

$S_1 = 2$

$S_2 = 2\frac{(6+1)}{2}$

$S_2 = 7$

ii) $S_1 = T_1 = 2$

$T_2 = 7 - 2$

$T_2 = 5$

$S_3 = \frac{3(10)}{2}$

$= 15$

$\therefore T_3 = 15 - 7$

$T_3 = 8$

hence, $T_1 = 2$, $T_2 = 5$, $T_3 = 8$.

iii) $T_n = a + (n-1)d$

$= 2 + (n-1)3$

$T_n = 3n - 1$

c) $y = x^3 - 3x^2$

$\frac{dy}{dx} = 3x^2 - 6x$

$\frac{d^2y}{dx^2} = 6x - 6$

at $\frac{d^2y}{dx^2} = 0$

$6x - 6 = 0$

$x = 1$

at $x = 1$, $y = -2$ $\quad P(1, -2)$

and $\frac{dy}{dx} = 3(1)^2 - 6(1)$

$m = -3$

$\therefore$ Eqn of tangent

$y - y_1 = m(x - x_1)$

$y + 2 = -3(x - 1)$

$y + 2 = -3x + 3$

$y = -3x + 1$

OR $3x + y - 1 = 0$

## Question 2

a) $4x^2 - 12x - 1 = 0$

i) $\alpha + \beta = \frac{-b}{a}$

$= \frac{12}{4}$

$= 3$

ii) $\alpha\beta = \frac{c}{a}$

$= \frac{-1}{4}$

iii) $5\alpha^2\beta + 5\alpha\beta^2$

$= 5\alpha\beta(\alpha + \beta)$

$= 5(-\frac{1}{4})(3)$

$= \frac{-15}{4}$

iv) $\frac{2}{\alpha^2} + \frac{2}{\beta^2} = \frac{2\beta^2 + 2\alpha^2}{\alpha^2\beta^2}$

$= \frac{2((\alpha+\beta)^2 - 2\alpha\beta)}{(\alpha\beta)^2}$

$= \frac{2[3^2 - 2(-\frac{1}{4})]}{(-\frac{1}{4})^2}$

$= 304$

b) i) $\Delta < 0$ for no real roots

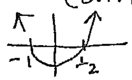
$b^2 - 4ac < 0$

$(4m)^2 + 4(2)(m-1) < 0$

$16m^2 + 8m - 8 < 0$

$2m^2 + m - 1 < 0$

$(2m-1)(m+1) < 0$




$-1 < m < \frac{1}{2}$

ii) let the roots be $\alpha$ and $\frac{1}{\alpha}$

$\alpha\beta = \frac{c}{a}$

$\alpha . \frac{1}{\alpha} = -\frac{(m-1)}{2}$

$1 = \frac{-m+1}{2}$

$2 = -m + 1$

$\therefore m = -1$

c) $2x^2 + 3x - 9 \equiv ax(x-1) + b(x-1) + c$

R.H.S $= ax^2 - ax + bx - b + c$

$= ax^2 + x(b-a) - b + c$

$\therefore a = 2$, $\quad b - a = 3$ $\quad -b + c =$

$b - 2 = 3$ $\quad -5 + c$

$b = 5$ $\quad c = -$

d) $f'(x)$ is positive

OR $f'(x) > 0$

and $f''(x)$ is negative

$f''(x) < 0$

## Question 3

a) i) $r = \frac{16}{12} = 1\frac{1}{3}$

$n = 8 \times 12 = 96$

$\therefore A_8 = 20000\left(1 + \frac{1\frac{1}{3}}{100}\right)^{96}$

$= \$71\ 326.94$

ii) $\text{Interest} = 71326.94 - 50\,000$

$= \$51\ 326.94$

iii) $I = \frac{Prt}{100}$

$51326.94 = \frac{20000 \times r \times 8}{100}$

$\frac{51326.94 \times 100}{20000 \times 8} = r$

$\therefore r = 32\%$ p.a.

b) $y = 7 + 4x^3 - 3x^4$

i) $\frac{dy}{dx} = 12x^2 - 12x^3$

stationary pts at $\frac{dy}{dx} = 0$

$12x^2 - 12x^3 = 0$

$12x^2(1 - x) = 0$

$\therefore$ at $x = 0$ , $x = 1$

$y = 7$ , $y = 8$

$P_1(0,7)$ & $P_2(1,8)$

ii) $\frac{dy}{dx} = 12x^2 - 12x^3$

$\frac{d^2y}{dx^2} = 24x - 36x^2$

at $x = 0$ , $\frac{d^2y}{dx^2} = 0$

Test concavity

| $x$ | $0^-$ | $0$ | $0^+ (\frac{1}{2})$ |
|---|---|---|---|
| $\frac{d^2y}{dx^2}$ | $-$ | $0$ | $+$ |

$\therefore$ since change in concavity

$\therefore (0,7)$ is a horizontal point of inflexion

at $x = 1$ , $\frac{d^2y}{dx^2} < 0$

$\therefore$ Maximum stationary point.

iii) $\frac{d^2y}{dx^2} = 0$

$24x - 36x^2 = 0$

$12x(2 - 3x) = 0$

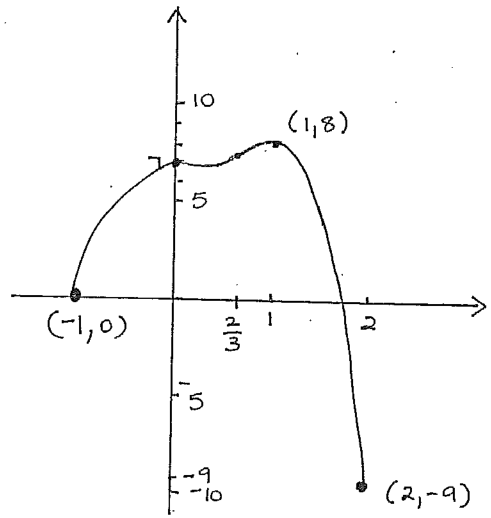
$x = 0$ $\quad x = \frac{2}{3}$

$y = 0$ $\quad y = 7\frac{16}{27}$

## Question 3 (con't)

iv) when $x = -1$, $y = 0$

when $x = 2$, $y = -9$




v) Concave up

$0 < x < \frac{2}{3}$

## Question 4

a) $ar^2 = -24 = T_3$

$ar^5 = 3 = T_6$

$\therefore \frac{r^5}{r^2} = \frac{3}{-24}$

$r^3 = -\frac{1}{8}$

$\therefore r = -\frac{1}{2}$

Since $|r| < 1$ $\therefore$ limiting sum exists.

$T_3 = ar^2 = -24$

$a(-\frac{1}{2})^2 = -24$

$a = -96$

$\therefore S_\infty = \frac{a}{1-r}$

$= \frac{-96}{1 + \frac{1}{2}}$

$S_\infty = -64$

b) $y = m(x-1)$ and $y = 2x^2$

$2x^2 = mx - m$

$2x^2 - mx + m = 0$

$a = 2,\ b = -m,\ c = m$

$\Delta = b^2 - 4ac$

$= (-m)^2 - 4(2)m$

$= m^2 - 8m$

Tangent at $\Delta = 0$ (equal roots)

$m^2 - 8m = 0$

$m(m - 8) = 0$

$m = 0$ or $m = 8$

c) $y = \sqrt{2x+1}$

$y = (2x+1)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}(2x+1)^{-1/2} \times 2$

$= \frac{1}{\sqrt{2x+1}}$

$\frac{d^2y}{dx^2} = -\frac{1}{2}(2x+1)^{-3/2} \times 2$

$= -\frac{1}{(2x+1)^{3/2}}$

## Question 4 (con't)

d) i)

Balance after 1 month $= \$15000 \times 1.01 - \$M$

ii) Balance after two months

$= (\$15000 \times 1.01 - \$M) \times 1.01 - \$M$

$= \$15000 \times 1.01^2 - \$M(1 + 1.01)$

Balance after three months

$= \$15000 \times 1.01^3 - \$M(1 + 1.01 + 1.01^2)$

Hence after n months

$= \$15000 \times 1.01^n - \$M(1 + 1.01 + 1.01^2 + \cdots + 1.01^{n-1})$

$= \$15000 \times 1.01^n - \$M\left(\frac{1.01^{36} - 1}{1.01 - 1}\right)$

iii) when $n = 36$, balance $= 0$

$\therefore \$15000 \times 1.01^{36} - \$M\left(\frac{1.01^{36} - 1}{1.01 - 1}\right) = 0$

$\$15000 \times 1.01^{36} = \$M\left(\frac{1.01^{36} - 1}{1.01 - 1}\right)$

$\frac{\$15000 \times 1.01^{36} \times 0.01}{1.01^{36} - 1} = \$M$

$\therefore \$M = \$498.21$

iv) Interest paid

$= \$498.21 \times 36 - \$15000$

$= \$2935.56$

v) Equivalent simple interest rate

$= \frac{2935.56}{15000} \times 100 \div 3$

$= 6.52\%$ p.a.

## Question 5

a) $P_n = 2^{n-1} + 5$

i) $P_1 = 2^0 + 5$

$P_1 = 6$

$P_7 = 2^6 + 5$

$= 69$

ii) $261 = 2^{n-1} + 5$

$256 = 2^{n-1}$

$2^8 = 2^{n-1}$

$n - 1 = 8$

$\therefore n = 9$

$\therefore$ The $9^{th}$ term is 261

b) $x^2 + px + 4 = 0$

$\alpha + \alpha + 3 = -p$

$2\alpha + 3 = -p \quad \ldots (1)$

$\alpha(\alpha + 3) = 4$

$\alpha^2 + 3\alpha - 4 = 0$

$(\alpha + 4)(\alpha - 1) = 0$

$\alpha = -4$ or $\alpha = 1$

$\therefore 2\alpha + 3 = -p$

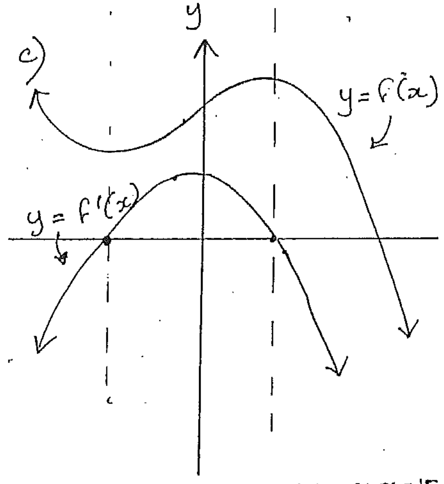
If $\alpha = -4$, $-8 + 3 = -p$

$\therefore p = 5$

If $\alpha = 1$, $2 + 3 = -p$

$\therefore p = -5$

c)



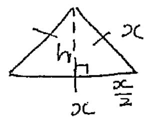

$x + y + x + x = 15$

d) i) Perimeter: $3x + y = 1$

$\therefore y = 1500 - 3x$

ii) Area of rectangle $= xy$

Area of Triangle $= \frac{1}{2} b.h.$




$= \frac{1}{2} . x . \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}x^2}{4}$

$h^2 = x^2 - \left(\frac{x}{2}\right)^2$

$h^2 = x^2 - \frac{x^2}{4}$

$h^2 = \frac{3x^2}{4}$

$\therefore h = \frac{\sqrt{3}x}{2}$

$\therefore$ Total area $= xy + \frac{\sqrt{3}}{4}$

$= x(1500 - 3x) + \frac{\sqrt{3}x}{4}$

$A = 1500x - 3x^2 + \frac{\sqrt{3}x^2}{4}$

$= 1500x - x^2\left(3 - \frac{\sqrt{3}}{4}\right)$

Question 5 (con't)

iii) $A = 1500x - x^2\left(3 - \frac{\sqrt{3}}{4}\right)$

$$\frac{dA}{dx} = 1500 - 2x\left(3 - \frac{\sqrt{3}}{4}\right)$$

$$\frac{d^2A}{dx^2} = -2\left(3 - \frac{\sqrt{3}}{4}\right) < 0$$

$\therefore$ only maximum area is possible through calculus.

$\frac{dA}{dx} = 0$ for max value.

$$1500 - 2x\left(3 - \frac{\sqrt{3}}{4}\right) = 0$$

$$2x\left(3 - \frac{\sqrt{3}}{4}\right) = 1500$$

$$x = \frac{1500}{2\left(3 - \frac{\sqrt{3}}{4}\right)}$$

$$\therefore x = 292 \cdot 2$$

$$y = 1500 - 3x$$

$$y = 1500 - 3(292 \cdot 2)$$

$$y = 623.4\text{m}$$

$\therefore$ Dimensions of rectangle is 292.2m wide and 623.4m length