South Sydney High School
Extension I Mathematics

Inequalities – Polynomials

Term: Term 1 Week 8
Time allowed: 80 minutes
Date: Wednesday 21st March 2012
Assessment: Extension I Mathematics

Question 1:

(a) Write $4.2\dot{5}\dot{1}$ as a mixed numeral show all working. **2**

(b) Simplify $9^{2x+1} \div 27^{3x}$ **2**

(c) Factorise

(i) $4x^2+12xy+9y^2$ **1**

(ii) $8x^3+27y^3$ **1**

(iii) Hence simplify $\dfrac{8x^3+27y^3}{4x-6y} \times \dfrac{4x^2-9y^2}{4x^2+12xy+9y^2}$ **1**

(d) Show that $\dfrac{3}{4-\sqrt{2}}+\dfrac{3}{4+\sqrt{2}}$ is rational. **2**

(e) If α, β and γ are the roots of $x^3-2x^2+4x+1=0$, evaluate $(\alpha+1)(\beta+1)(\gamma+1)$ **3**

Question 2:

(a) A polynomial is defined by $P(x)=-x^3+x^2+14x-24$.

(i) Show that $(x-2)$ is a factor of $P(x)$. **1**

(ii) Fully factorise and solve $P(x)=0$. **2**

(iii) Graph the polynomial $P(x)=-x^3+x^2+14x-24$ **2**

(iv) Hence solve $\dfrac{3x}{x-2} \le x+6$ **3**

(b) (i) By completing the square show that the roots of the equation $2x^2-5x-12=0$ are $x=-\dfrac{3}{2}$ and $x=4$. **2**

(ii) By using the substitution $u=2^x$, solve the equation $2^{x+1}-5=\dfrac{6}{2^{x-1}}$ **2**

Question 3:

(a) Solve $|2y+7|+|y-2| \ge 10$. **3**

(b) If one root of the equation $36x^3-36x^2-x+k=0$ is equal to the difference between the other two roots, find:

(i) The roots of the equation. **3**

(ii) The value of k. **1**

(c) (i) Show that $(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=\alpha^3+\beta^3$ **2**

(ii) The roots of the equation $x^2+(x+1)^2-k=0$ are α and β.

(α) Prove that $\alpha^3+\beta^3=\dfrac{1}{2}(1-3k)$. **1**

(β) Find, in terms of k, a quadratic equation in x, whose roots are α^3 and β^3. **2**

South Sydney High School
Extension I Mathematics

Inequalities - Polynomials

Term: Term 1 Week 8 | **Date:** Wednesday 29th March 2012
Time allowed: 60 minutes | **Assessment:** Extension I Mathematics

Question 1:

(a) Write $4.2\dot{5}\dot{1}$ as a mixed numeral show all working. **2**

Let $x = 4.2\dot{5}\dot{1}$ ---- (1)

$10x = 42.51\dot{5}\dot{1}$ ---- (2)

$1000x = 4251.\dot{5}\dot{1}$ ---- (3)

Now equation (3)-(2)

$1000x = 4251.\dot{5}\dot{1}$ ---- (3)

$10x = 42.51\dot{5}\dot{1}$ ---- (2)

$990x = 4209$

$$\therefore x = \frac{4209}{990} = 4\frac{83}{330}$$

(b) Simplify $9^{2x+1} \div 27^{3x}$ **2**

$9^{2x+1} \div 27^{3x}$

$= (3^2)^{2x+1} \div (3^3)^{3x}$

$= 3^{4x+2} \div 3^{9x}$

$= 3^{-5x+2}$

(c) Factorise

(i) $4x^2 + 12xy + 9y^2$ **1**

$= (2x+3y)^2$

(ii) $8x^3 + 27y^3$ **1**

$= (2x+3y)(4x^2 - 6xy + 9y^2)$

(iii) Hence simplify $\dfrac{8x^3+27y^3}{4x-6y} \times \dfrac{4x^2-9y^2}{4x^2+12xy+9y^2}$ **1**

$$= \frac{(2x+3y)(4x^2-6xy+9y^2)}{2(2x-3y)} \times \frac{(2x-3y)(2x+3y)}{(2x+3y)^2}$$

$$= \frac{(4x^2-6xy+9y^2)}{2}$$

(d) Show that $\dfrac{3}{4-\sqrt{2}} + \dfrac{3}{4+\sqrt{2}}$ is rational. **2**

$$= \frac{3(4+\sqrt{2}) + 3(4-\sqrt{2})}{(4-\sqrt{2})(4+\sqrt{2})}$$

$$= \frac{12+3\sqrt{2}+12-3\sqrt{2}}{16-2} = \frac{24}{14} = \frac{12}{7} \quad \text{ie Rational.}$$

(e) If α, β and γ are the roots of $x^3 - 2x^2 + 4x + 1 = 0$, evaluate $(\alpha+1)(\beta+1)(\gamma+1)$ **3**

$x^3 - 2x^2 + 4x + 1 = 0$

$\alpha + \beta + \gamma = \dfrac{-b}{a} = 2$

$\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{c}{a} = 4$

$\alpha\beta\gamma = \dfrac{-d}{a} = -1$

$(\alpha+1)(\beta+1)(\gamma+1) = \alpha\beta\gamma + \alpha\beta + \beta\gamma + \gamma\alpha + \alpha + \beta + \gamma + 1$

$= -1 + 4 + 2 + 1$

$= 6$

Question 2:

(a) A polynomial is defined by $P(x)=-x^3+x^2+14x-24$.

(i) Show that $(x-2)$ is a factor of $P(x)$. **1**

$P(x)=-x^3+x^2+14x-24.$

$P(2)=-2^3+2^2+14\times2-24.$

$=0$

$\therefore$ $(x-2)$ is a factor.

(ii) Fully factorise and solve $P(x)=0$. **2**

$$\begin{array}{r} -x^2-x+12 \\ x-2\overline{)-x^3+x^2+14x-24} \\ \underline{-x^3+2x^2} \\ -x^2+14x \\ \underline{-x^2+2x} \\ 12x-24 \\ \underline{12x-24} \\ 0 \end{array}$$

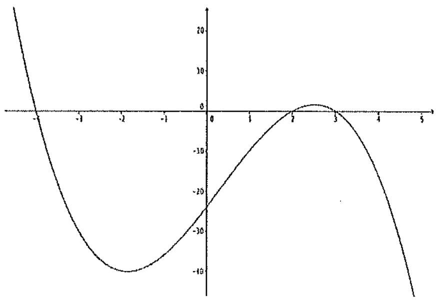
$\therefore P(x)=(x-2)(-x^2-x+12)$

$=-(x-2)(x^2+x-12)$

$=-(x-2)(x+4)(x-3)$

$\therefore x=2,\ -4,\ 3$

(iii) Graph the polynomial $P(x)=-x^3+x^2+14x-24$ **2**



(iv) Hence solve $\dfrac{3x}{x-2}\le x+6$ **3**

Multiply each term by $(x-2)^2$

$$3x(x-2)\le(x+6)(x-2)^2$$

$$3x(x-2)-(x+6)(x-2)^2\le0$$

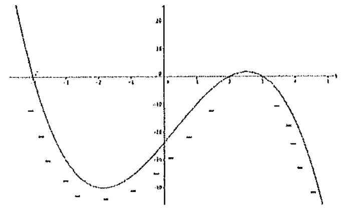
$$(x-2)\left[3x-(x+6)(x-2)\right]\le0$$

$$(x-2)\left[3x-(x^2+4x-12)\right]\le0$$

$$(x-2)(3x-x^2-4x+12)\le0$$

$$(x-2)(-x^2-x+12)\le0$$

$$-(x-2)(x+4)(x-3)\le0$$



$-4\le x<2\ ,\ x\geq 3$

(b) (i) By completing the square show that the roots of the equation $2x^2-5x-12=0$ are $x=-\dfrac{3}{2}$ and $x=4$. **2**

$$2x^2-5x-12=0$$

$$x^2-\frac{5}{2}x=6$$

$$x^2-\frac{5}{2}x+\left(-\frac{5}{2}\times\frac{1}{2}\right)^2=6+\left(-\frac{5}{2}\times\frac{1}{2}\right)^2$$

$$\left(x-\frac{5}{4}\right)^2=6+\frac{25}{16}$$

$$\left(x-\frac{5}{4}\right)^2=\frac{121}{16}$$

$$x-\frac{5}{4}=\pm\frac{11}{4}$$

$$\therefore x=\frac{5\pm11}{4}$$

$$\therefore x=\frac{5+11}{4}=\frac{16}{4}\quad\text{or}\quad\therefore x=\frac{5-11}{4}=\frac{-6}{4}$$

$$\therefore x=4\quad\text{or}\quad\therefore x=-\frac{3}{2}$$

(ii) By using the substitution $u=2^x$, solve the equation $2^{x+1}-5=\frac{6}{2^{x-1}}$ 2

$$2^{x+1}-5=\frac{6}{2^{x-1}}$$

$$2(2^x)-5=\frac{6}{(2^x)2^{-1}}$$

$$2(2^x)-5=\frac{12}{2^x}$$

$$2(2^x)^2-5(2^x)-12=0$$

Let $2^x = u$

$$2u^2-5u-12=0$$

From previous question

$u=\frac{-3}{2}$ or $u = 4$

i.e $2^x=\frac{-3}{2}$ or $2^x=4$

No solution or $x=2$

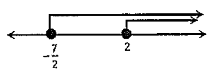
Question 3:

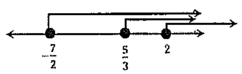
(a) Solve $|2y+7|+|y-2|\geq 10$. 3

Case 1:

$2y+7\geq 0$ and $y-2\geq 0$

$y\geq -\frac{7}{2}$ and $y\geq 2$





$2y+7+y-2\geq 10$

$3y\geq 5$

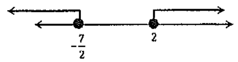
$y\geq\frac{5}{3}$

$\therefore\ y\geq 2$

Case 2:

$2y+7\leq 0$ and $y-2\geq 0$

$y\leq -\frac{7}{2}$ and $y\geq 2$



No Solution.

Case 3:

$2y+7\leq 0$ and $y-2\leq 0$

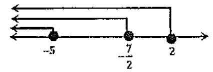
$y\leq -\frac{7}{2}$ and $y\leq 2$

$$-1(2y+7)+-(y-2)\geq 10$$

$$-2y-7-y+2\geq 10$$

$$-3y\geq 15$$

$$y\leq -5$$



$\therefore y\leq -5$

Case 4:

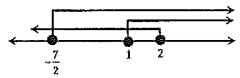
$2y+7\geq 0$ and $y-2\leq 0$

$y\geq -\frac{7}{2}$ and $y\leq 2$

$$2y+7+-(y-2)\geq 10$$

$$2y+7-y+2\geq 10$$

$$y\geq 1$$



$\therefore 1\leq y\leq 2$

Final Answer

$\therefore y\leq -5$ or $y\geq 1$

(b) If one root of the equation $36x^3-36x^2-x+k=0$ is equal to the difference between the other two roots, find:

(i) The roots of the equation. **3**

Let the roots be $\alpha, \beta, \alpha-\beta$

$$\text{Sum of roots} = \alpha+\beta+\alpha-\beta=\frac{-b}{a}=\frac{36}{36}=1$$

i.e. $2\alpha=1$

$$\alpha=\frac{1}{2}$$

Product of roots, 2 at a time $=\alpha\beta+\alpha^2-\alpha\beta+\alpha\beta-\beta^2$

$$=\alpha^2+\alpha\beta-\beta^2=\frac{c}{a}=\frac{-1}{36}$$

$$\text{Sub } \alpha=\frac{1}{2}: \frac{1}{4}+\frac{\beta}{2}-\beta^2=\frac{-1}{36}$$

$$9+18\beta-36\beta^2=-1$$
$$36\beta^2-18\beta-10=0$$
$$18\beta^2-9\beta-5=0$$
$$(6\beta-5)(3\beta+1)=0$$
$$6\beta-5=0 \text{ or } 3\beta+1=0$$
$$\beta=\frac{5}{6}, \beta=-\frac{1}{3}$$

$$\therefore \alpha-\beta=\frac{1}{2}-\frac{5}{6}=\frac{-1}{3} \text{ or } \alpha-\beta=\frac{1}{2}-(\frac{-1}{3})=\frac{5}{6}$$

$\therefore$ Roots are $\frac{1}{2}, \frac{5}{6}, \frac{-1}{3}$

(ii) The value of k. **1**

Product of all 3 roots $\alpha\beta(\alpha-\beta)=\frac{-d}{a}$

$$\frac{1}{2}\times\frac{5}{6}(-\frac{1}{3})=\frac{-K}{36}$$
$$K=5$$

(c) (i) Show that $(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=\alpha^3+\beta^3$ **2**

$$(\alpha+\beta)^3=(\alpha+\beta)(\alpha+\beta)^2$$
$$=(\alpha+\beta)(\alpha^2+2\alpha\beta+\beta)^2$$
$$=\alpha^3+2\alpha^2\beta+\alpha\beta^2+\alpha^2\beta+2\alpha\beta^2+\beta^3$$
$$=\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3$$
$$(\alpha+\beta)^3-3\alpha^2\beta-3\alpha\beta^2=\alpha^3+\beta^3$$
$$\therefore (\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=\alpha^3+\beta^3$$

(ii) The roots of the equation $x^2+(x+1)^2=k$ are α and β.

(α) Prove that $\alpha^3+\beta^3=\frac{1}{2}(1-3k)$. **1**

$$2x^2+2x+1-k=0$$

$$a=2, \quad b=2, \quad c=1-k$$
$$\therefore \alpha+\beta=-\frac{b}{a}=\frac{-2}{2}=-1 \quad \text{and} \quad \alpha\beta=\frac{c}{a}=\frac{1}{2}(1-k)$$

$$\therefore \alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$$
$$=(-1)^3-3\times\frac{1}{2}(1-k)\times-1$$
$$=-1+\frac{3}{2}(1-k)$$
$$=-1+\frac{3}{2}-\frac{3k}{2}$$
$$=\frac{1}{2}-\frac{3k}{2}$$
$$=\frac{1}{2}(1-3k)$$

(β) Find, in terms of k, a quadratic equation in x, whose roots are α^3 and β^3. **2**

$$\alpha^3\beta^3=(\alpha\beta)^3=\frac{1}{8}(1-k)^3$$

The quadratic equation with roots α^3 and β^3 is

$x^2-(\textit{sum of new roots})x+(\textit{product of new roots})$

$$x^2-\frac{1}{2}(1-3k)x+\frac{1}{8}(1-k)^3=0$$
$$8x^2-4(1-3k)x+(1-k)^3=0$$