St George Girls High School

Year 12

Mid-HSC Course Examination

2012

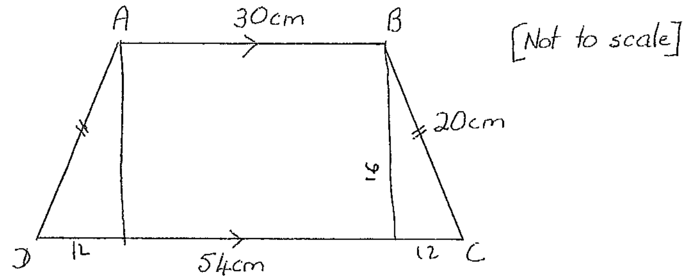
Mathematics Extension 1

General Instructions
- Working time – 90 minutes
- Reading time – 5 minutes
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Total marks – 72
- Attempt Questions 1 – 6
- All questions are of equal value

Question 1 – (12 marks) – (Start A New Booklet) Marks

a) In your examination booklet write down the letter that corresponds to the correct answer.

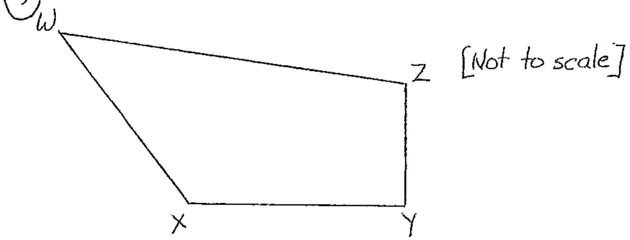


$ABCD$ is an isosceles trapezium with measurements as shown on the diagram. The area of $ABCD$ is: 1

A. 124 cm² B. 672 cm² C. 840 cm² D. 864 cm²

b) The size of each interior angle of a regular polygon is 156°. Find the number of sides of this polygon. (Working must be shown to gain full marks) 2

c)



$WXYZ$ is a quadrilateral in which: 3

$WX = XY = 2.YZ$
$< WXZ = < XYZ = 90°$

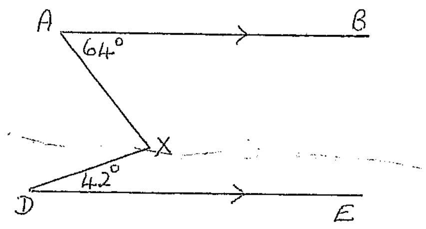
(i) Copy this diagram into your answer booklet, showing the above information.

(ii) Prove that $WZ = 3.YZ$

Question 1 – (cont'd) Marks

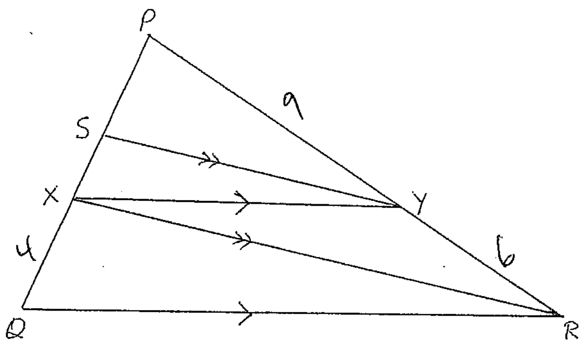
d) Show that if one angle of a triangle is equal to the sum of the other two angles then the triangle is right angled. 2

e) 1



Find the size of reflex $< AXD$ (Reasons not necessary).

f)



PQR is a triangle.
X and Y are points on PQ and PR respectively such that $XY \| QR$.
S is a point on PQ such that $SY \| XR$.

If $PY = 9, YR = 6$ and $XQ = 4$ find:

(i) PX and hence 1

(ii) SX 2

You must give clear geometric reasons for your answer.

Question 2 – (12 marks) – (Start A New Booklet) Marks

a) **In your examination booklet write down the letter that corresponds to the correct answer.**

The exact value of $\sin A$ is $\frac{1}{\sqrt{5}}$. The exact value of $\sin 2A$ could be: 1

A. $\frac{1}{5}$ B. $\frac{2}{\sqrt{5}}$ C. $\frac{3}{\sqrt{5}}$ D. $\frac{-4}{5}$

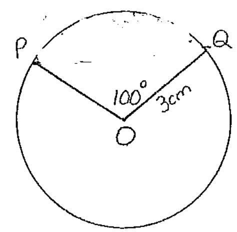
b) Express $\frac{5\pi}{9}$ radians in degrees. 1

c) Find the exact value of $\sec\frac{5\pi}{6}$ 1

d) Find the equation of the tangent to the curve $y = 3 + \tan 3x$ at the point where $x = \frac{\pi}{4}$ 3

e) Find $\lim_{x \to 0} \frac{\tan\frac{3x}{4}}{4x}$ (You must justify your answer). 2

f)

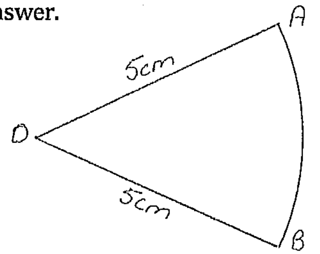


(i) Find the exact value of the area of minor sector POQ. 2

(ii) Find the area of the major segment cut off by chord PQ. (Answer correct to 2 decimal places). 2

Question 3 – (12 marks) – (Start A New Booklet) Marks

a) In your examination booklet write down the letter that corresponds to the correct answer.



The area of sector AOB is 25cm². The perimeter of the sector is: 1

A. 10 cm B. 12 cm C. 15 cm D. 20 cm

b) Prove that $\dfrac{\sin 2x}{1+\cos 2x} \equiv \tan x$ 2

c) Find:

(i) $\displaystyle\int \frac{\cos x}{1+\sin x}\,dx$ 1

(ii) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{3\pi}{2}} \sin x\ dx$ 2

(iii) $\displaystyle\int_{0}^{\frac{\pi}{3}} sin^2\,2x\,dx$ 2

d) (i) Show that $\tan^3 x = \sec^2 x\ \tan x - \tan x$ 1

(ii) Hence find $\displaystyle\int \tan^3 x\,dx$ 3

Question 4 – (12 marks) – (Start A New Booklet) Marks

In your examination booklet write down the letter that corresponds to the correct answer for Parts a) and b).

a) For the function $y = \tan^{-1}\sqrt{1-x^2}$ the largest possible range is 1

A. $0 \le y \le \frac{\pi}{4}$

B. $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$

C. $-\frac{\pi}{4} < y < \frac{\pi}{2}$

D. all real numbers

b) The area bounded by the curve $y = \tan x$, the x-axis and the line $x = \frac{\pi}{4}$ is rotated about the x-axis. The volume (in cubic units) of the solid generated is 1

A. $\log_e\sqrt{2}$ B. $1 - \frac{\pi}{4}$ C. $\pi - \frac{\pi^2}{4}$ D. π

c) (i) Show that the curves $y = \sin 2x$ and $y = \cos x$ intersect at the point $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$ 1

(ii) Find θ, the acute angle between the curves at this point of intersection (Answer to the nearest degree) 4

d) (i) Write down a general solution for the equation $\cos\theta = \cos\frac{\pi}{3}$ 1

(ii) Hence find all solutions of $\cos\theta = \cos\frac{\pi}{3}$ for $-\pi \le \theta \le 2\pi$ 1

e) If $f(x) = x\sin^{-1}2x$ find the exact value of $f'\left(\frac{1}{2\sqrt{2}}\right)$ 3

Question 5 – (12 marks) – (Start A New Booklet) Marks

a) **In your examination booklet write down the letter that corresponds to the correct answer.**

The curve $y = 8x^3$ is reflected in the line $y = x$. The equation of the new curve is: 1

A. $y = \frac{1}{8x^3}$ B. $y = -8x^3$ C. $y = \frac{\sqrt[3]{x}}{2}$ D. $y = 2x$

b) Given that $f(x) = 3\sin^{-1}\left(\frac{x-1}{2}\right)$

(i) write down the domain and range of $y = f(x)$ 2

(ii) draw a neat sketch of $y = f(x)$, showing all important features. (Your sketch should be approximately $\frac{1}{3}$ of a page). 2

c) Find the equation of the normal to the curve $y = \tan^{-1}(x+1)$ at the point where $x = 0$ 3

d) Find the following:

(i) $\int \frac{1}{9+16x^2}\,dx$ 1

(ii) $\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{\sqrt{4-9x^2}}\,dx$ [exact value] 3

Question 6 – (12 marks) – (Start A New Booklet) Marks

a) **In your examination booklet write down the letter that corresponds to the correct answer.**

The area under the curve $y = \frac{1}{\sqrt{4-x^2}}$ between $x = 0$, $x = 1$ and the x-axis is: 1

A. $\sqrt{3} - 1$ B. $2 - \sqrt{3}$ C. $\frac{\pi}{6}$ D. $\frac{\pi}{3}$

b) Given that $\int_2^{2\sqrt{2}} \frac{-1}{\sqrt{16-x^2}}\,dx = k\pi$ find the exact value of k 3

c) (i) Show that $\frac{d}{dx}\left\{x\tan^{-1}3x - \frac{1}{6}\log_e(1+9x^2)\right\} = \tan^{-1}3x$ 1

(ii) Hence find the exact value of $\int_0^{\frac{1}{\sqrt{3}}} \tan^{-1}3x\,dx$ 2

d) (i) Find $\frac{d}{dx}(\sin^{-1}x + \cos^{-1}x)$. 1

(ii) Hence, or otherwise, prove that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ 2

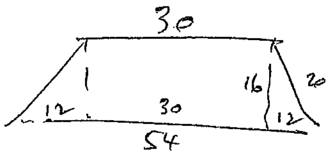
(iii) Sketch the curve $y = \sin^{-1}x + \cos^{-1}x$ 2

End of Paper

2012 Ext 1 Yr 12 Mid Course

Q1

a)



$A = \frac{1}{2}(16)(30+54)$

$= 8(84)$

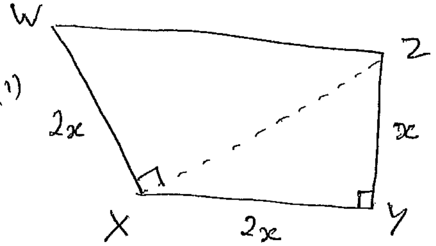
$= 672$

$\therefore B$

b) Ext angle $= 24^\circ$

$360 \div 24 = 15$ $\therefore$ 15 sides.

c)



(i)

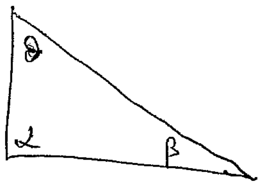
(ii) Let $YZ = x$

$XZ = \sqrt{5}\,x$

$WZ = 3x$

$= 3\,YZ$

d)



Let the angles be θ, α, β

where $\alpha = \theta + \beta$

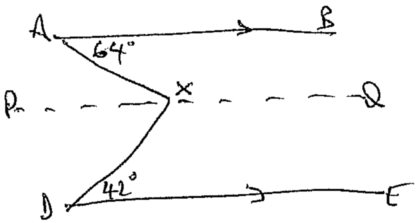
$\theta + \beta + \alpha = 180^\circ$ (angle sum of a triangle)

$\theta + \beta + \theta + \beta = 180^\circ$

$\theta + \beta = 90^\circ$

$\therefore$ triangle is right.

e)



Construct $PQ \parallel AB$ passing through X

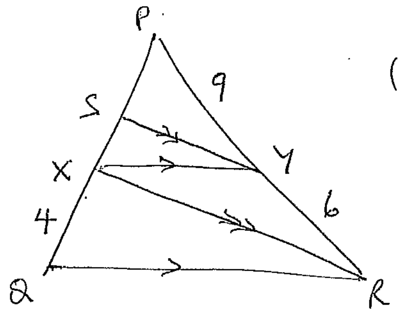
$\angle AXQ = 116^\circ$ (co-interior angles ; $AB \parallel XQ$)

$\angle DXQ = 138^\circ$ (co-interior angles ; $XQ \parallel DE$)

$\therefore$ reflex $\angle AXD = 116 + 138$

$= 254^\circ$

f)



(i) $\frac{9}{6} = \frac{PX}{4}$ (ratio of intercepts on parallel lines)

$6PX = 36$

$PX = 6$

(ii) $\frac{PS}{SX} = \frac{9}{6}$ (ratio of intercepts on parallel lines)

$6PS = 9SX \longrightarrow 2PS = 3SX$

$PS + SX = 6$

$\frac{3}{2}SX + SX = 6$

$5SX = 12$

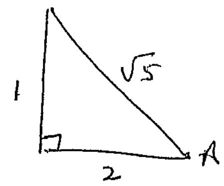
$SX = \frac{12}{5}$

Q2

a) $\sin 2A = 2\sin A\cos A$

$= 2 \times \frac{1}{\sqrt{5}} \times \frac{2}{\sqrt{5}}$

$= \frac{4}{5}$



as $\cos A = \pm\frac{2}{\sqrt{5}}$

$\therefore \sin 2A = -\frac{4}{5}$

D

b) $\frac{5\pi}{9} = \frac{5}{9} \times 180^\circ$

$= 100^\circ$

c) $\sec \frac{5\pi}{6} = \frac{1}{\cos \frac{5\pi}{6}}$

$= \frac{1}{-\cos \frac{\pi}{6}}$

$= -\frac{2}{\sqrt{3}}$

d) $y = 3 + \tan 3x$ at $x = \frac{\pi}{4}$

$y = 3 + -1$

$= 2$

$\frac{dy}{dx} = 3 \sec^2 3x$

at $x = \frac{\pi}{4}$ $\frac{dy}{dx} = 3(\sqrt{2})^2$

$= 6$

Equation of tangent

$y - 2 = 6(x - \frac{\pi}{4})$

e) $\lim_{x \to 0} \frac{\tan \frac{3x}{4}}{4x} = \lim_{x \to 0} \frac{\tan \frac{3x}{4}}{\frac{3x}{4} \cdot \frac{16}{3}}$

$= \frac{3}{16} \lim_{x \to 0} \frac{\tan \frac{3x}{4}}{\frac{3x}{4}}$

$= \frac{3}{16}$

f) i) Area segment POQ $= \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \times 3^2 \times \frac{5\pi}{9}$

$= \frac{5\pi}{2}$

Area is $\frac{5\pi}{2}$ cm^2

ii) Area of major segment

$= \frac{1}{2} \times 3^2 \times \frac{13\pi}{9} + \frac{1}{2} \times 3^2 \times \sin \frac{5\pi}{9}$

$= \frac{13\pi}{2} + \frac{9}{2} \sin \frac{5\pi}{9}$

$= 24.85$ (2 dpl)

Area of major segment 24.85 cm^2

Q3

a) $\frac{1}{2} r^2 \theta = 25$

$\theta = 2$

$P = 5 + 5 + 5 \times 2$

$= 20$

$\therefore D$

b) $\frac{\sin 2x}{1 + \cos 2x} \equiv \tan x$

LHS $= \frac{\sin 2x}{1 + \cos 2x}$

$= \frac{2 \sin x \cos x}{1 + \cos^2 x - \sin^2 x}$

$= \frac{2 \sin x \cos x}{1 + 2\cos^2 x - 1}$

$= \frac{2 \sin x \cos x}{2 \cos^2 x}$

$= \frac{\sin x}{\cos x}$

$= \tan x$

$=$ RHS.

c) i) $\displaystyle\int \frac{\cos x}{1+\sin x}\,dx = \ln(1+\sin x) + c$

ii) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{3\pi}{2}} \sin x\,dx = \Big[-\cos x\Big]_{\frac{\pi}{6}}^{\frac{3\pi}{2}}$

$= -\left[\cos\frac{3\pi}{2} - \cos\frac{\pi}{6}\right]$

$= -\left[0 - \frac{\sqrt{3}}{2}\right]$

$= \frac{\sqrt{3}}{2}$

iii) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^2 2x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{3}} (1-\cos 4x)\,dx$

$= \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{3}}$

$= \frac{1}{2}\left[\left(\frac{\pi}{3} - \frac{1}{4}\sin\frac{4\pi}{3}\right) - (0-0)\right]$

$= \frac{\pi}{6} + \frac{\sqrt{3}}{16}$

d) i) $\tan^3 x = \sec^2 x \tan x - \tan x$

RHS $= \sec^2 x \tan x - \tan x$

$= \tan x(\sec^2 x - 1)$

$= \tan x(\tan^2 x)$

$= \tan^3 x$

$=$ LHS

ii) $\displaystyle\int \tan^3 x\,dx = \int (\sec^2 x \tan x - \tan x)\,dx$

$= \frac{1}{2}\tan^2 x + \ln\cos x + c$

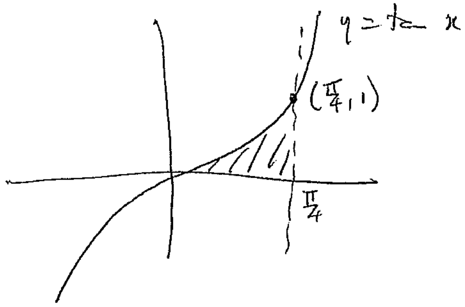
Q4

a) $y = \tan^{-1}\sqrt{1-x^2}$

$-1 \le x \le 1$

$\therefore\ 0 \le y \le \frac{\pi}{4}$ $\therefore$ A.

b)



$V = \pi\displaystyle\int_0^{\frac{\pi}{4}} [\tan x]^2\,dx$

$= \pi\displaystyle\int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\,dx$

$= \pi\left[\tan x - x\right]_0^{\frac{\pi}{4}}$

$= \pi\left[\left(1-\frac{\pi}{4}\right) - (0-0)\right]$

$= \pi\left[1 - \frac{\pi}{4}\right]$ $\therefore$ C

c) i) $y = \sin 2x$

$y = \cos x$

At point of intersection

$\sin 2x = \cos x$

$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ or $2\sin x = 1$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ $\quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$

At $x = \frac{\pi}{6}$ $\quad y = \frac{\sqrt{3}}{2}$

OR when $x = \frac{\pi}{6}$

$y = \sin 2\left(\frac{\pi}{6}\right)$

$= \sin\frac{\pi}{3}$

$= \frac{\sqrt{3}}{2}$

$y = \cos\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2}$

$\therefore \left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$ is a point of intersection

c) ii) $y = \sin 2x$ $\qquad$ $y = \cos x$

$\frac{dy}{dx} = 2\cos 2x$ $\qquad$ $\frac{dy}{dx} = -\sin x$

at $x = \frac{\pi}{6}$ $\quad$ $\frac{dy}{dx} = 2 \times \frac{1}{2}$ $\qquad$ $= -\frac{1}{2}$

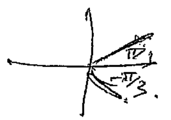
$= 1$

$$\tan\theta = \frac{1-(-\frac{1}{2})}{1+1\times(-\frac{1}{2})}$$

$= 3$

$\theta = 72^\circ$ (to nearest degree)

d) i) $\cos\theta = \cos\frac{\pi}{3}$



$\theta = \frac{\pi}{3} + 2\pi n$ or $-\frac{\pi}{3} + 2\pi n$ for some integer n.

ii) $\theta = -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$.

e) $f(x) = x\sin^{-1} 2x$

$$f'(x) = \sin^{-1} 2x + \frac{2x}{\sqrt{1-(2x)^2}}.$$

$$f'\left(\frac{1}{2\sqrt{2}}\right) = \sin^{-1}\frac{1}{\sqrt{2}} + \frac{\frac{1}{\sqrt{2}}}{\sqrt{1-(\frac{1}{\sqrt{2}})^2}}$$

$= \frac{\pi}{4} + 1$

Q5

a) $y = 8x^3$

For reflection in $y = x$

$x = 8y^3$

$y^3 = \frac{x}{8}$

$y = \frac{\sqrt[3]{x}}{2}$ $\qquad$ i.e. $\quad$ C

b) $f(x) = 3\sin^{-1}\left(\frac{x-1}{2}\right)$

i) D: $-1 \le \frac{x-1}{2} \le 1$

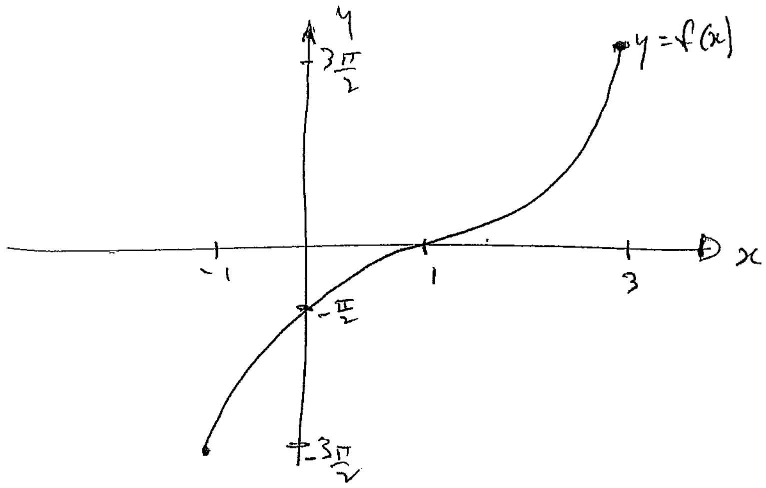
$-2 \le x-1 \le 2$

$-1 \le x \le 3$

R: $-\frac{\pi}{2} \le \frac{y}{3} \le \frac{\pi}{2}$

$-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

ii)



c) $y = \tan^{-1}(x+1)$

$$\frac{dy}{dx} = \frac{1}{1+(x+1)^2}$$

at $x=0$ $\quad \frac{dy}{dx} = \frac{1}{1+1^2}$

$$= \frac{1}{2}$$

$\therefore$ gradient of normal -2.

$$y - \frac{\pi}{4} = -2(x-0)$$

$$y = -2x + \frac{\pi}{4}$$

d) i) $\int \frac{dx}{9+16x^2}\ dx$

$$= \int \frac{dx}{16(\frac{9}{16}+x^2)}$$

$$= \frac{1}{16}\int \frac{dx}{\frac{9}{16}+x^2} \qquad a = \frac{3}{4}$$

$$= \frac{1}{16}\ \frac{1}{3/4}\tan^{-1}\frac{x}{3/4} + c$$

$$= \frac{1}{12}\tan^{-1}\frac{4x}{3} + c$$

ii) $\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{\sqrt{4-9x^2}}\ dx = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{\sqrt{9(\frac{4}{9}-x^2)}}\ dx$

$$= \frac{1}{3}\int_0^{1/\sqrt{3}} \frac{1}{\sqrt{(2/3)^2-x^2}}$$

$$= \frac{1}{3}\left[\sin^{-1}\frac{x}{2/3}\right]_0^{1/\sqrt{3}}$$

$$= \frac{1}{3}\left[\sin^{-1}\frac{3x}{2}\right]_0^{1/\sqrt{3}}$$

$$= \frac{1}{3}\left[\sin^{-1}\frac{3/\sqrt{3}}{2} - \sin^{-1}0\right]$$

$$= \frac{1}{3}\sin^{-1}\frac{\sqrt{3}}{2}$$

$$= \frac{1}{3}\cdot\frac{\pi}{3}$$

$$= \frac{\pi}{9}$$

Q6

a) $\int_0^1 \frac{1}{\sqrt{4-x^2}}\ dx = \left[\sin^{-1}\frac{x}{2}\right]_0^1$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1}0$$

$$= \frac{\pi}{6}$$

$\therefore C$

b) $\int_2^{2\sqrt{2}} \frac{-1}{\sqrt{16-x^2}}\ dx = -\left[\sin^{-1}\frac{x}{4}\right]_2^{2\sqrt{2}}$

$$= -\left[\sin^{-1}\frac{2\sqrt{2}}{4} - \sin^{-1}\frac{2}{4}\right]$$

$$= -\left[\sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}\frac{1}{2}\right]$$

$$= -\left[\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= -\frac{\pi}{12}$$

$\therefore k = -\frac{1}{12}$

c) i) Show

$$\frac{d}{dx}\left[x\tan^{-1}3x - \frac{1}{6}\ln(1+9x^2)\right]$$

$$= x \cdot \frac{3}{1+9x^2} + \tan^{-1}3x - \frac{1}{6}\cdot\frac{1}{1+9x^2}\cdot 18x$$

$$= \frac{3x}{1+9x^2} + \tan^{-1}3x - \frac{3x}{1+9x^2}$$

$$= \tan^{-1}3x$$

ii) $\displaystyle\int_0^{\frac{1}{\sqrt{3}}} \tan^{-1}3x \, dx$

$$= \left[x\tan^{-1}3x - \frac{1}{6}\ln(1+9x^2)\right]_0^{1/\sqrt{3}}$$

$$= \left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{3}{\sqrt{3}} - \frac{1}{6}\ln\left(1+9\left(\tfrac{1}{3}\right)\right)\right] - \left[0 - \frac{1}{6}\ln 1\right]$$

$$= \frac{1}{\sqrt{3}}\cdot\frac{\pi}{3} - \frac{1}{6}\ln 4 - 0$$

$$= \frac{\sqrt{3}\,\pi}{9} - \frac{1}{6}\ln 4$$

d) $\displaystyle\frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}}$

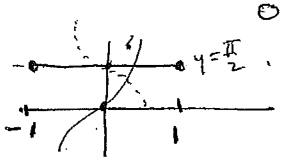
$= 0$

ii) $\sin^{-1}x + \cos^{-1}x = c$ as $\frac{d}{dx}[\;] = 0$.

when $x = 0$ $\quad \sin^{-1}0 + \cos^{-1}0 = c$

$0 + \frac{\pi}{2} = c \quad \therefore c = \frac{\pi}{2}$

e)



NB. $\frac{dy}{dx}$ defined $-1 < x < 1$

when $x = -1$ $\quad \sin^{-1}(-1) + \cos^{-1}(-1) = \frac{\pi}{2}$